Towards a Higher-Order Solver for Aerodynamics Using a Discontinuous Galerkin Discretization

David L. Darmofal
Aerospace Computational Design Lab
Massachusetts Institute of Technology

November 12, 2004

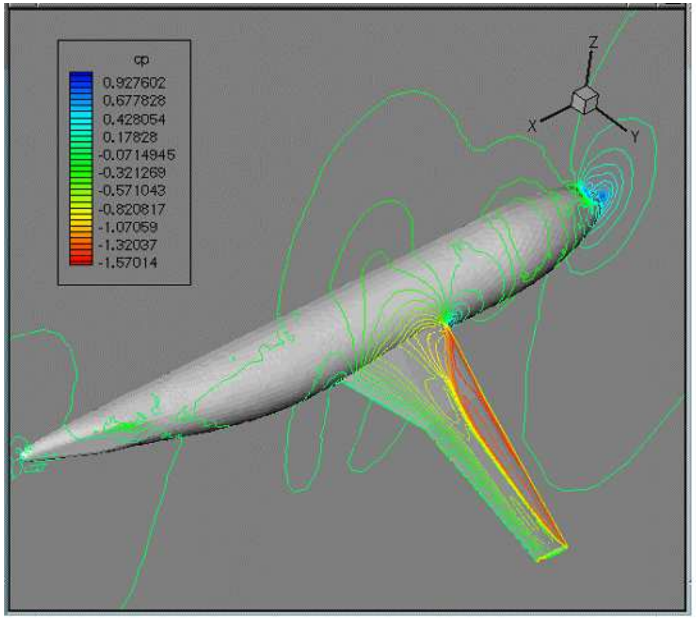
Overview

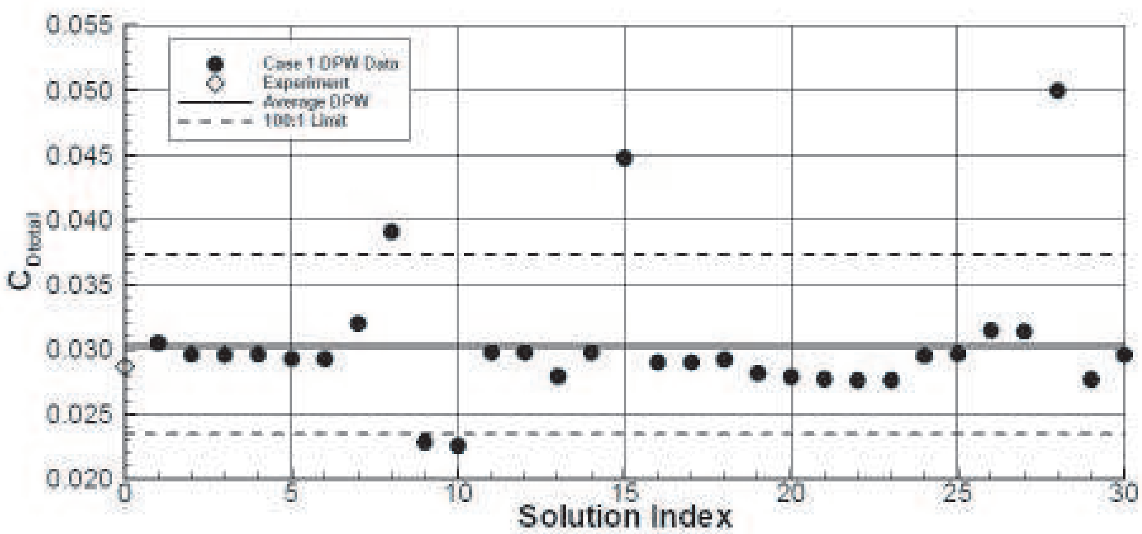
- Motivation: Why another CFD algorithm for aerodynamics?
- Finite volume methods for hyperbolic conservation laws
- Discontinuous Galerkin (DG) for hyperbolic conservation laws
- DG for elliptic problems
- p-multigrid for higher-order DG discretizations
- Conclusions and future work

Motivation for higher order

- Higher-order methods are critical for simulation of unsteady flows with multiple scales, e.g.:
  - Applications of DNS, LES, or DES
  - Acoustics
- Even in aerodynamics, higher-order methods may offer benefits:
  - Existing 'industrial-strength' methods largely based on finite-volume with at best second order accuracy
  - Questions exist whether current discretizations are capable of achieving desired accuracy levels in practical time

AIAA Drag Prediction Workshop



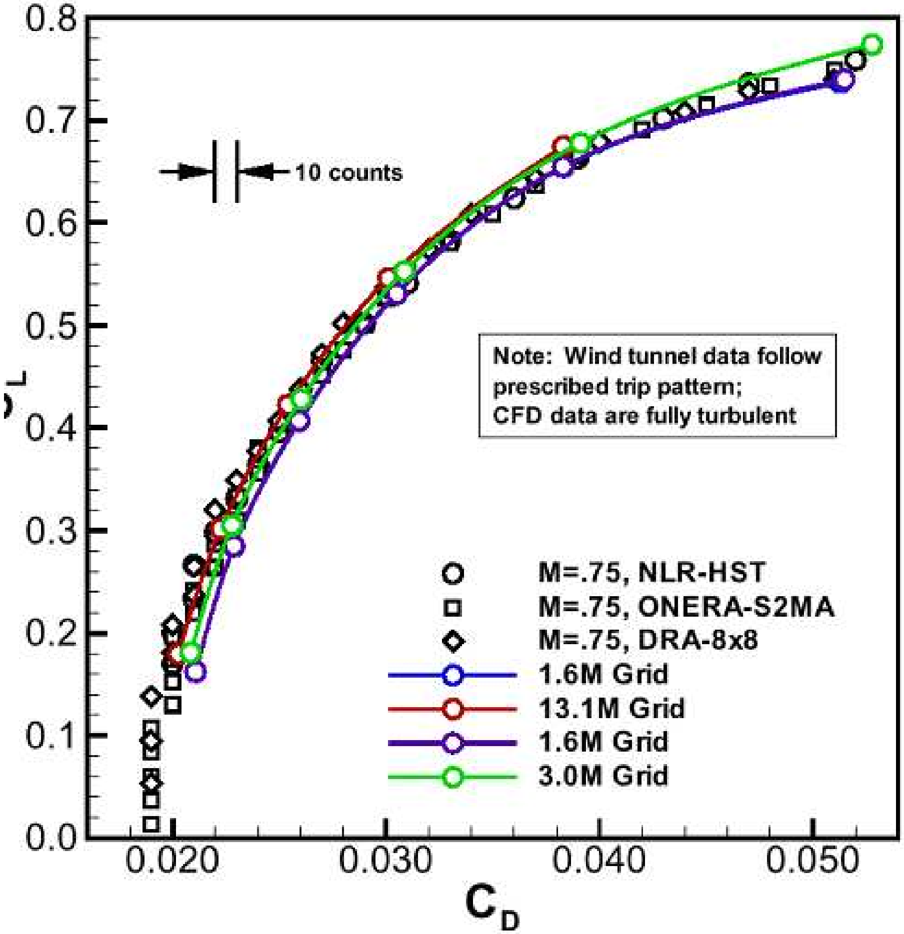


Transonic drag prediction studied for a wing-body configuration

Excluding outliers, uncertainty in C_D is 40 counts of drag amoung 18 different simulations

Note: 1 count of drag is the equivalent of about 5-10 passengers

AIAA Drag Prediction Workshop: Mesh Studies



Simulations by Mavriplis

At low C_L, discretization error is about 10 drag counts from coarse to fine mesh

At high C_L, discretization error is about 20 drag counts.

Experimental uncertainty is about 10 drag counts throughout polar, but can be reduced to about 1 drag count with better corrections.

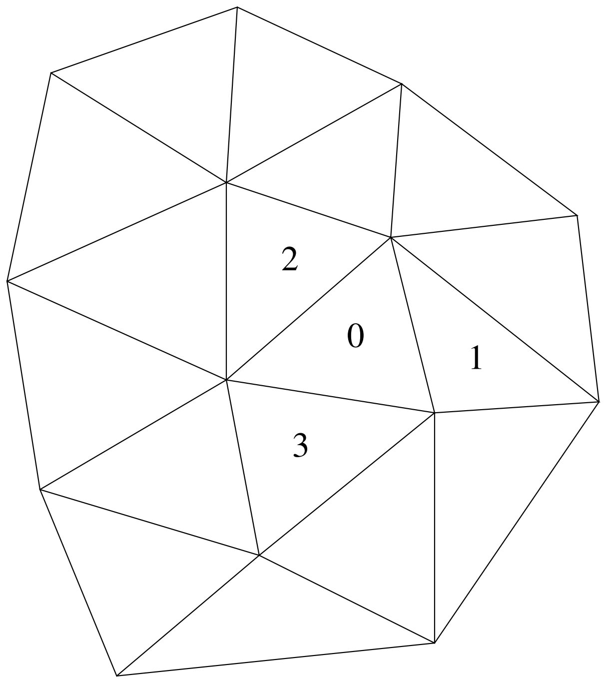
Higher-order CFD Project

Team Goal: To improve the aerothermal design process for complex 3D configurations by significantly reducing the time from geometry to solution at engineering-required accuracy using high-order adaptive methods

- Students
  - Garrett Barter (shock limiting)
  - Jean-Baptiste Brachet (shock limiting)
  - Mike Brasher (visualization)
  - Tan Bui (unsteady aero/structures)
  - Krzysztof Fidkowski (multigrid solver)
  - James Lu (optimization and adaptation)
  - Todd Oliver (viscous discretization)
  - Mike Park (3-D meshing)

- Advisors
  - David Darmofal
  - Robert Haimes
  - Jaime Peraire
  - Karen Wilcox

First-order Accurate Finite Volume



In each triangle, assume $\mathbf{u}$ is constant.

Apply conservation law on triangle:

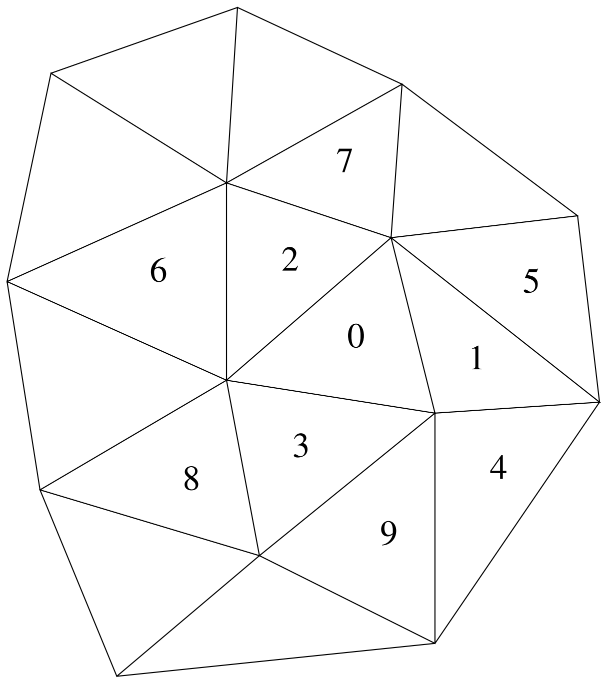
$$\frac{d\mathbf{u}_0}{dt}A_0 + \sum_{k=1}^{3} \int_{0k} \mathcal{H}_i(\mathbf{u}_0, \mathbf{u}_k, \hat{\mathbf{n}}_{0k})\, ds = 0$$

$\mathcal{H}_i(\mathbf{u}_L, \mathbf{u}_R, \hat{\mathbf{n}}_{LR})$ is flux function that determines inviscid flux in $\hat{\mathbf{n}}_{LR}$ direction from left and right states, $\mathbf{u}_L$ and $\mathbf{u}_R$.

Example flux functions: Godunov, Roe, Osher, Van Leer, Lax-Friedrichs, etc.

This discretization has a solution error which is $O(h)$ where h is mesh size.

Second-order Accurate Finite Volume



In each triangle, reconstruct a linear solution, $\tilde{\mathbf{u}}$, using neighboring averages:

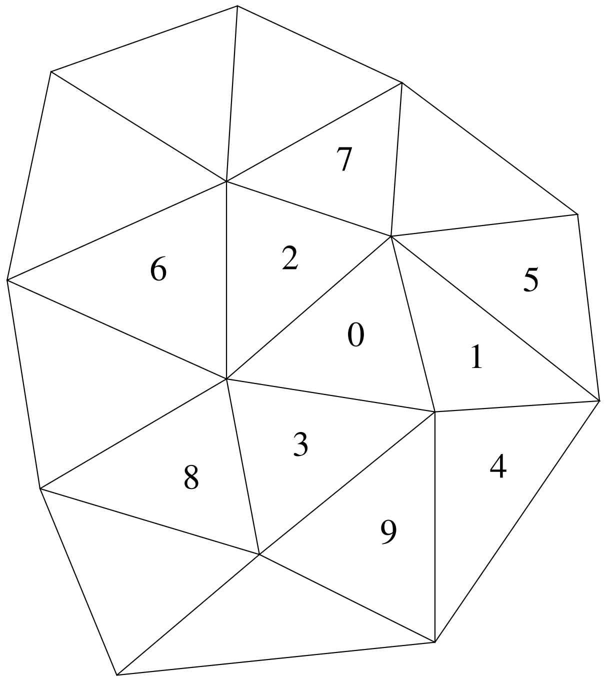
$$\begin{aligned}\tilde{\mathbf{u}}_0 &\equiv \mathbf{u}_0 + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla \mathbf{u}_0, \\ \nabla \mathbf{u}_0 &\equiv \nabla \mathbf{u}_0 \left(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\right).\end{aligned}$$

Apply conservation law on triangle:

$$\frac{d\mathbf{u}_0}{dt} A_0 + \sum_{k=1}^{3} \int_{0k} \mathcal{H}_i(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_k, \hat{\mathbf{n}}_{0k})\, ds = 0$$

On smooth meshes and flows, solution error is $O(h^2)$.

Pros/Cons of Higher-order Finite Volume



+ Increased accuracy on given mesh without additional degrees of freedom

– Difficulty in achieving higher-order on unstructured meshes and near boundaries

– Single stage, local iterative methods (e.g. Jacobi) are not stable for higher order (Godunov's theorem)

– Matrix fill-in increased resulting in high-memory requirements

Previous Work in DG

- Extensive work on DG for hyperbolic equations
  - Bassi and Rebay (1997)
  - Cockburn and Shu (1998, 2001)
  - Karniadakis et al. (1998, 1999)
- More recently work begun on elliptic equations
  - Bassi and Rebay (1997,1998)
  - Cockburn and Shu (1998, 2001)
  - Baumann and Oden (1997)
  - Brezzi et al. (1997)
- Only Bassi and Rebay have published RANS results (1997, 2003)

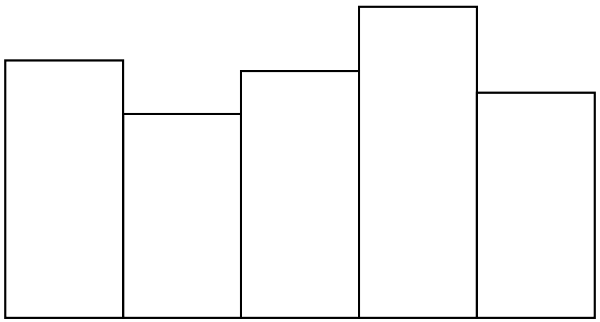
Discontinuous Polynomial Basis

- Triangulate domain Ω into non-overlapping elements $\kappa \in T_h$
- Define function space: Element-wise discontinuous polynomials of degree p

$$\mathcal{V}_h^p = \{\mathbf{v} \in L^2(\Omega) : \mathbf{v}|_\kappa \in P^p(\kappa) : \forall \kappa \in T_h\}$$

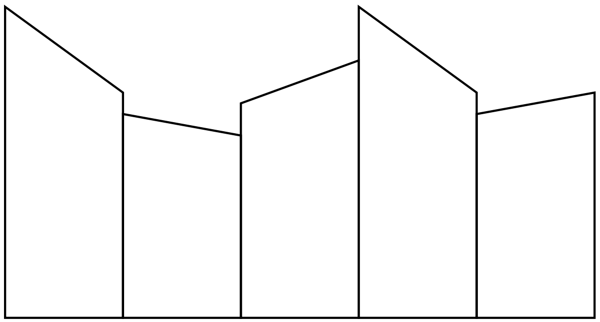
Example of One-Dimensional Bases

$p = 0$ basis



1 DOF/element

$p = 1$ basis



2 DOF/element

DG for Hyperbolic Conservation Laws: Derivation

Find $\mathbf{u}_h \in \mathcal{V}_h^p$, such that $\forall \kappa \in T_h$, $\forall \mathbf{v}_h \in \mathcal{V}_h^p$:

$$\int_\kappa \mathbf{v}_h^T (\mathbf{u}_h)_t \, d\mathbf{x} - \int_\kappa \nabla \mathbf{v}_h^T \cdot \mathcal{F}_i \, d\mathbf{x}$$

$$+ \int_{\partial\kappa \backslash \partial\Omega} \mathbf{v}_h^{+T} \mathcal{H}_i(\mathbf{u}_h^+, \mathbf{u}_h^-, \hat{\mathbf{n}}) ds + \int_{\partial\kappa \cap \partial\Omega} \mathbf{v}_h^{+T} \mathcal{H}_i^b(\mathbf{u}_h^+, \mathbf{u}_h^b, \hat{\mathbf{n}}) \, ds = 0.$$

Boundary conditions enforced weakly through $\mathcal{H}_i^b(\mathbf{u}_h^+, \mathbf{u}_h^b, \hat{\mathbf{n}})$ where $\mathbf{u}_h^b$ is determined from desired boundary conditions and outgoing characteristics.

For smooth problems, the error of this scheme is expected to be $O(h^{p+1})$.

Relationship of DG to other methods

- For $p = 0$ discretization, DG reduces to:

$$(\mathbf{u}_\kappa)_t A_\kappa + \int_{\partial\kappa\backslash\partial\Omega} \mathcal{H}_i(\mathbf{u}_h^+, \mathbf{u}_h^-, \hat{\mathbf{n}}) ds \int_{\partial\kappa\cap\partial\Omega} \mathcal{H}_i^b(\mathbf{u}_h^+, \mathbf{u}_h^b, \hat{\mathbf{n}})\, ds = 0.$$

- Thus, $p = 0$ DG is identical to first-order finite volume.

- For $p > 0$, DG can be intrepreted as a moment method.

- Moment methods for hyperbolic problems were first suggested by Van Leer (1977) and then developed for the Euler equations by Allmaras (1987, 1989) and later Holt (1992).

Stability of Local Iterative Methods for DG

An elemental block Jacobi iterative method to solve this problem is,

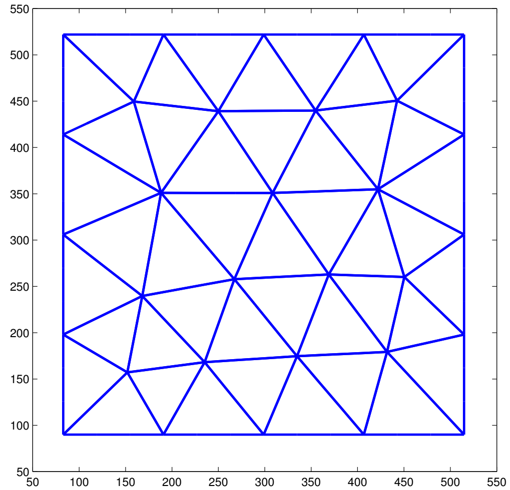
$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \omega \left(\partial \mathbf{R}_j / \partial \mathbf{u}_j\right)^{-1} \mathbf{R}_j(\mathbf{u}).$$

where $\partial \mathbf{R}_j / \partial \mathbf{u}_j$ is the diagonal block for the element j.

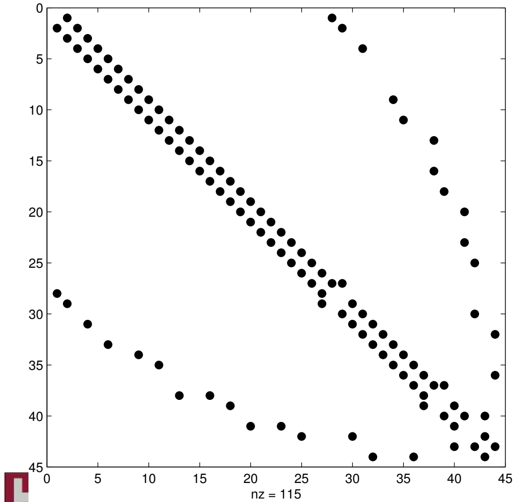
For 1-D hyperbolic systems, the eigenvalues of the higher-order modes are all collocated $\Rightarrow$ p-independent convergence.

For multiple dimensions, elemental block Jacobi is stable independent of p when $0 < \omega < 1$.

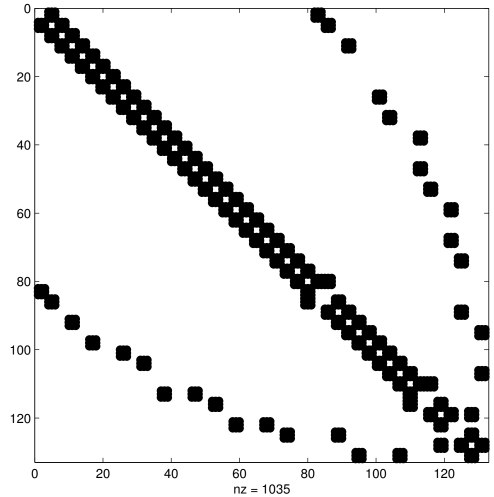
Matrix Fill for Higher-order DG



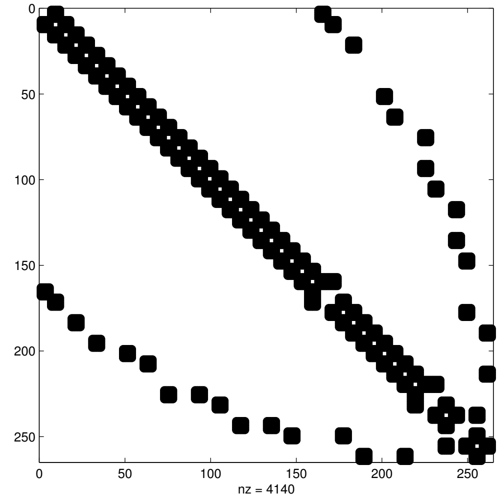
First-order ($p = 0$)



Second-order ($p = 1$)



Third-order ($p = 2$)



DG for Elliptic Operators: First Attempt

- Model problem for viscous terms of N-S: 1-D, scalar Poisson's equation

$$-u_{xx} = f \quad \text{on} \quad [-1, 1]$$

- Proceed as for Euler:
  - Triangulate domain into non-overlapping elements $\kappa \in T_h$
  - Define solution and test function space $\mathcal{V}_h^p$
- Discrete formulation: Find $u_h \in \mathcal{V}_h^p$ such that $\forall v_h \in \mathcal{V}_h^p$,

$$\sum_{\kappa \in T_h} \left\{ - \left[v_h \widehat{u_x} \right]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} + \int_\kappa (v_h)_x (u_h)_x dx \right\} = \sum_{\kappa \in T_h} \left\{ \int_\kappa v_h f dx \right\}$$

- Need to define $\widehat{u_x}$

DG for Elliptic Operators: First Attempt

- No upwinding mechanism $\Rightarrow$ choose central flux

$$\widehat{u_x} = \frac{1}{2}((u_h)_x^+ + (u_h)_x^-)$$

- Discrete formulation becomes: Find $u_h \in \mathcal{V}_h^p$ such that $\forall v_h \in \mathcal{V}_h^p$,

$$\sum_{\kappa \in T_h} \Big\{ - \Big[\frac{1}{2} v_h((u_h)_x^+ + (u_h)_x^-)\Big]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} + \int_\kappa (v_h)_x (u_h)_x dx \Big\} = \sum_{\kappa \in T_h} \Big\{ \int_\kappa v_h f dx \Big\}$$

- PROBLEM: Scheme is inconsistent! Constants can be added to all elements without altering residual.

First Order System Approach (Bassi & Rebay)

- Introduce new variable, $q = u_x$, such that

$$\begin{aligned} -q_x &= f \\ q - u_x &= 0 \end{aligned}$$

- Discrete formulation: Find $u_h \in \mathcal{V}_h^p$ and $q_h \in \mathcal{V}_h^p$ such that $\forall v_h \in \mathcal{V}_h^p$ and $\forall \tau_h \in \mathcal{V}_h^p$,

$$\begin{aligned} \sum_{\kappa \in T_h} \left\{ -\left[v_h \widehat{q} \right]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} + \int_\kappa (v_h)_x q_h dx \right\} - \sum_{\kappa \in T_h} \left\{ \int_\kappa v_h f dx \right\} &= 0 \\ \sum_{\kappa \in T_h} \left\{ \int_\kappa \tau_h q_h dx + \int_\kappa (\tau_h)_x u_h dx - \left[\tau_h \widehat{u} \right]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} \right\} &= 0 \end{aligned}$$

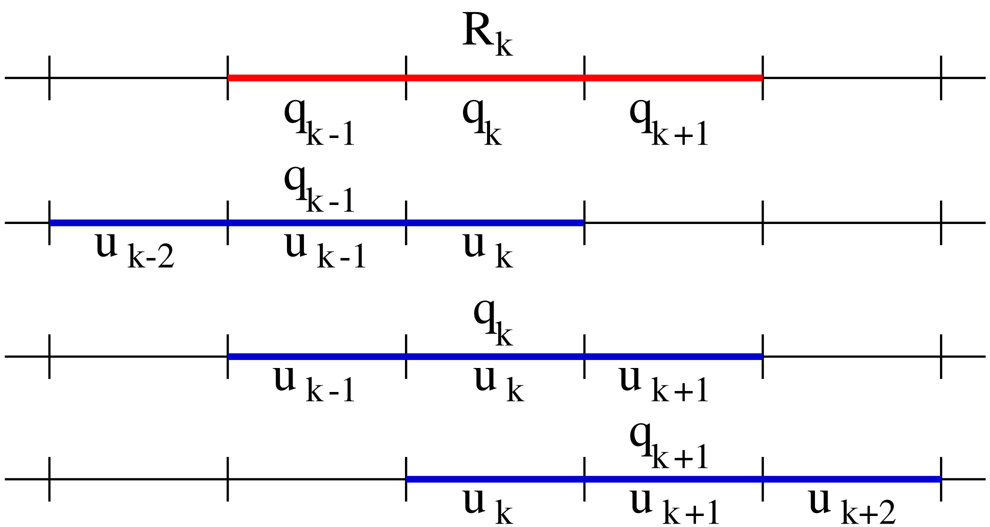
- Need to choose $\widehat{q}$ and $\widehat{u}$

First Method of Bassi & Rebay (BR1)

- No upwinding mechanism $\Rightarrow$ choose central fluxes

$$\widehat{u} = \frac{1}{2}(u_h^+ + u_h^-); \quad \widehat{q} = \frac{1}{2}(q_h^+ + q_h^-)$$

- Sub-optimal order of accuracy for odd p
- Stencil no longer compact



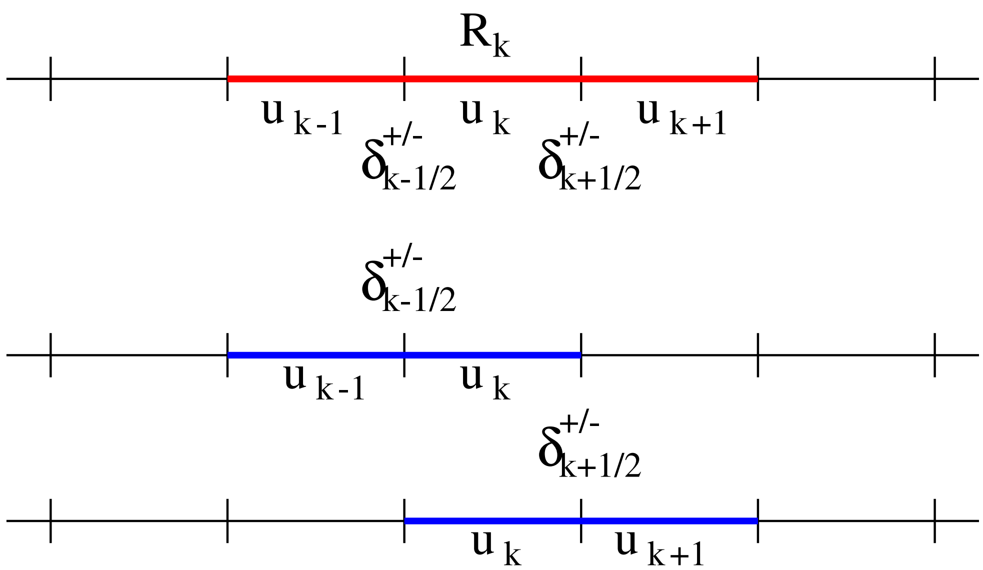
Second Method of Bassi & Rebay (BR2)

BR2 achieves optimial $p+1$ accuracy and has compact stencil:

$$\{s\} = \frac{1}{2}\left(s^{+} + s-\right); \quad [\![s]\!] = s^{+} - s^{-}$$

$$\widehat{u} = \{u_h\}; \quad \widehat{q} = \{(u_h)_x\} - \eta_f\{\delta_f\}$$

Where $\delta_f \in \mathcal{V}_h^p, \quad \forall \tau_h \in \mathcal{V}_h^p: \quad \int_{\kappa^\pm} \tau_h \delta_f^{\pm} dx = \left([\![u_h]\!]\{\tau_h\}\right)_f$



Iterative Solution of Higher-order DG (Fidkowski & Darmofal, 2004)

- Use a preconditioned iterative scheme to drive $\mathbf{R}(\mathbf{u}_h^n) \rightarrow 0$:

$$\mathbf{u}_h^{n+1} = \mathbf{u}_h^n - \mathbf{P}^{-1}\mathbf{R}(\mathbf{u}_h^n)$$

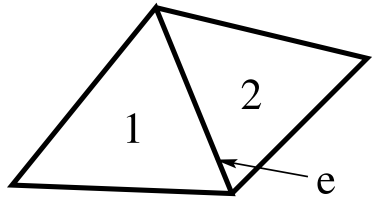
- Elemental line preconditioner: $\mathbf{P} = \mathbf{M}_{line}$
- Motivation: Transport of information in Navier-Stokes equations characterized by strong (anisotropic) coupling
  - Inviscid regions: Information follows characteristic directions set by convection
  - Boundary layers/wakes: Diffusion effects can be as strong if grid is well-resolved.
- Lines of elements from using an element-to-element coupling measure.

Connectivity Criteria

- Measure of influence based on $p = 0$ discretization of scalar transport equation

$$\nabla \cdot (\rho \vec{u} \phi) - \nabla \cdot (\mu \nabla \phi) = 0$$

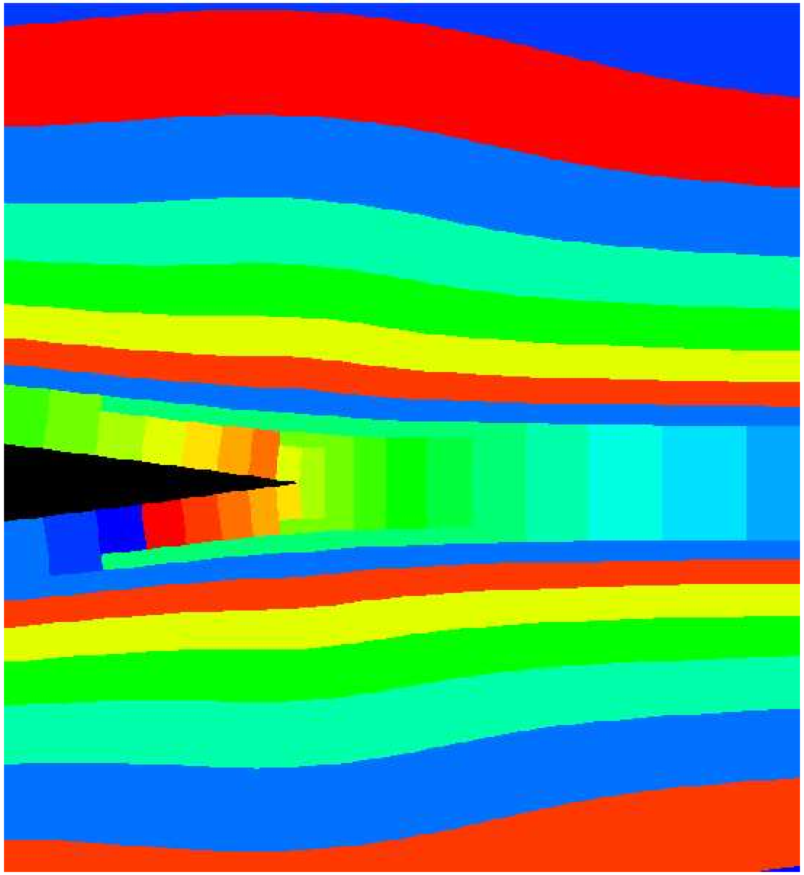
- $\rho \vec{u}$ and μ taken from current solution
- At each edge, compute off-diagonal components of Jacobian for adjoining elements

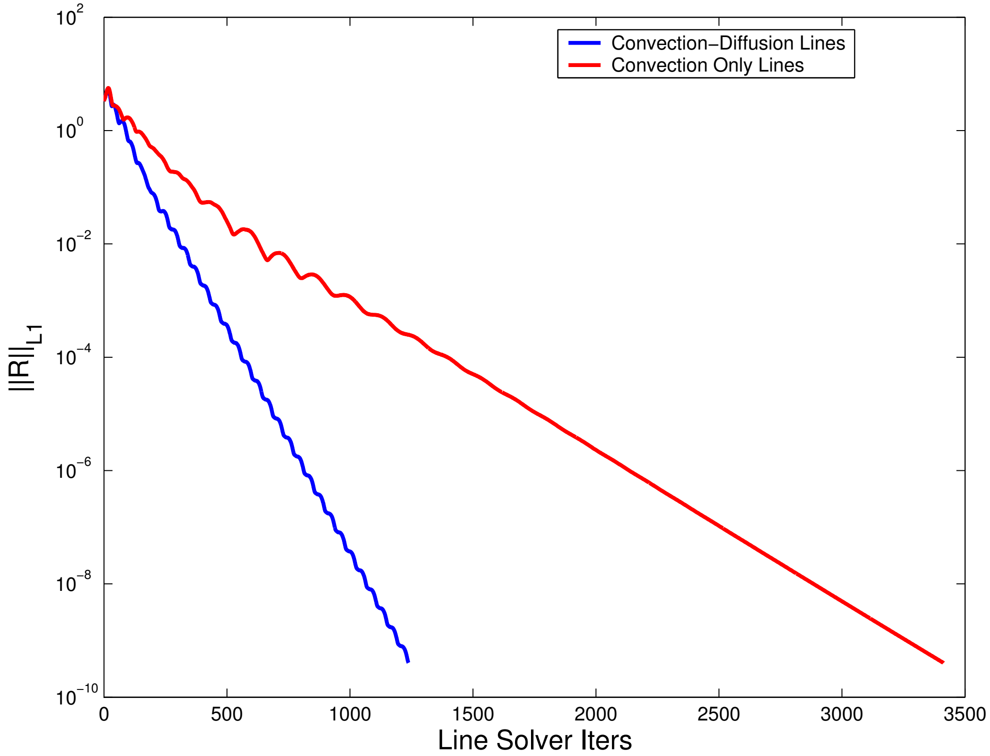


- Connectivity given by maximum absolute value

$$C_e = \max\left(\left| \frac{\partial R_1}{\partial \phi_2} \right|, \left| \frac{\partial R_2}{\partial \phi_1} \right| \right)$$

Example Lines and Performance



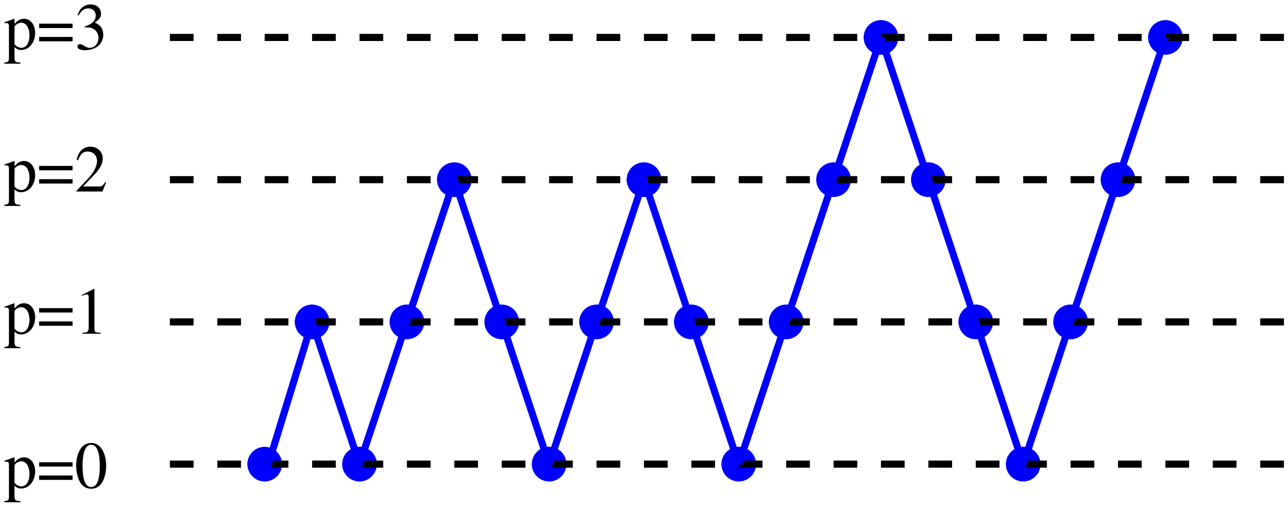


p-Multigrid: Motivation

- Observation: Smoothers are inefficient at eliminating low frequency error modes on fine level
- h-Multigrid
  - Spatially coarse grid used to correct solution on fine grid
  - Grid coarsening is complex on unstructured meshes
- p-Multigrid (Ronquist & Patera, Helenbrook et al., Fidkowski & Darmofal)
  - Low order ($p-1$) approximation used to correct high order (p) solution
  - Natural implementation in DG FEM discretization on unstructured meshes

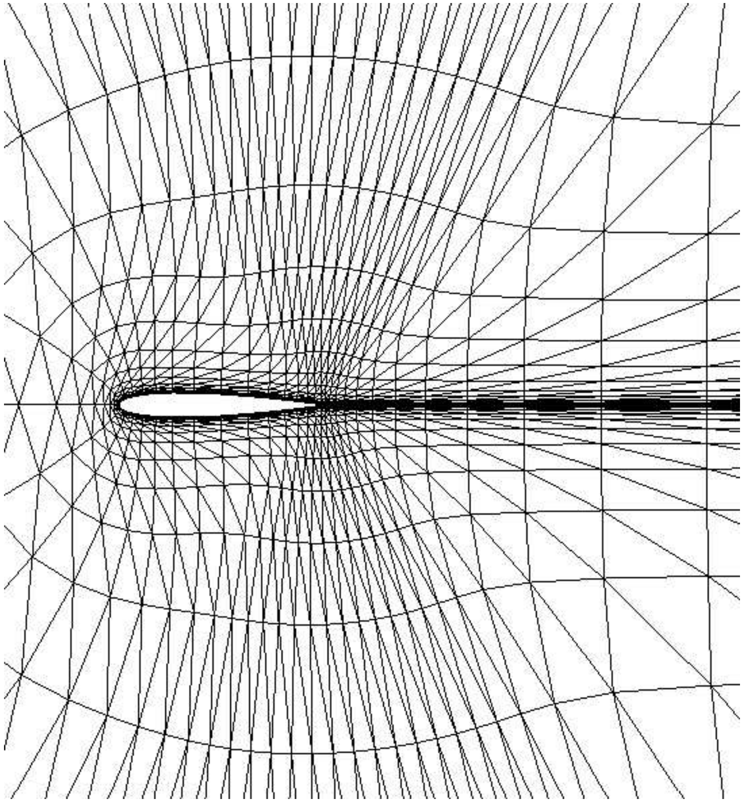
p-Multigrid: Full Multigrid

- Full Approximation Scheme (FAS) used
- Line solver used as smoother

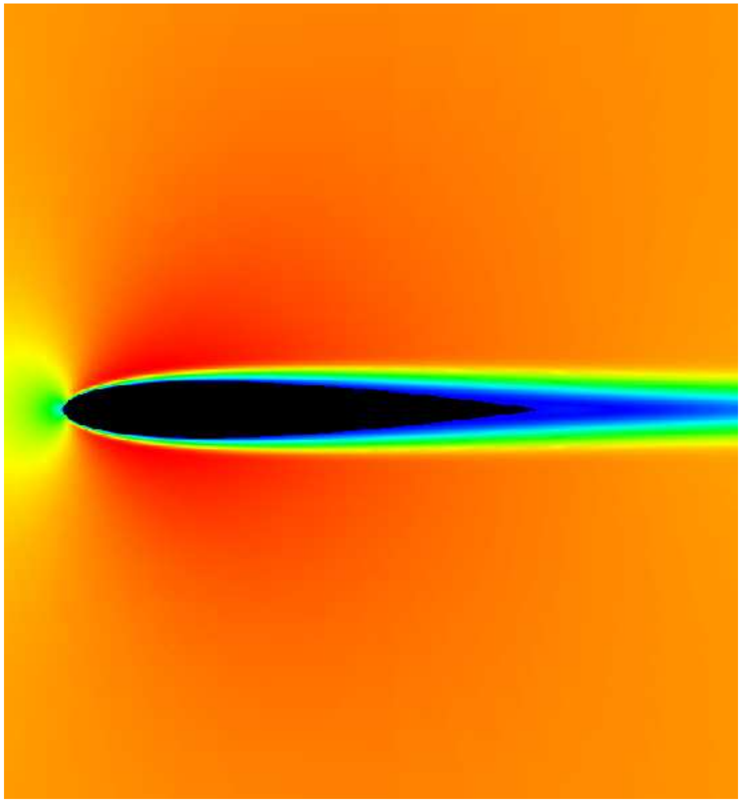


NACA 0012 Test Case

$M = 0.5$, $Re = 5000$, $\alpha = 0$
Grids are from Swanson at NASA Langley



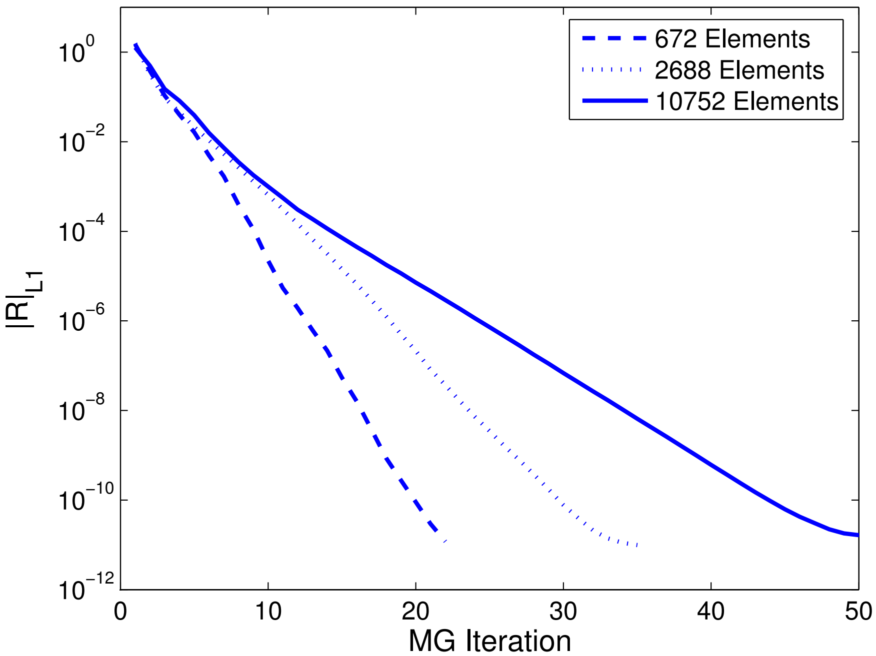
2112 element mesh



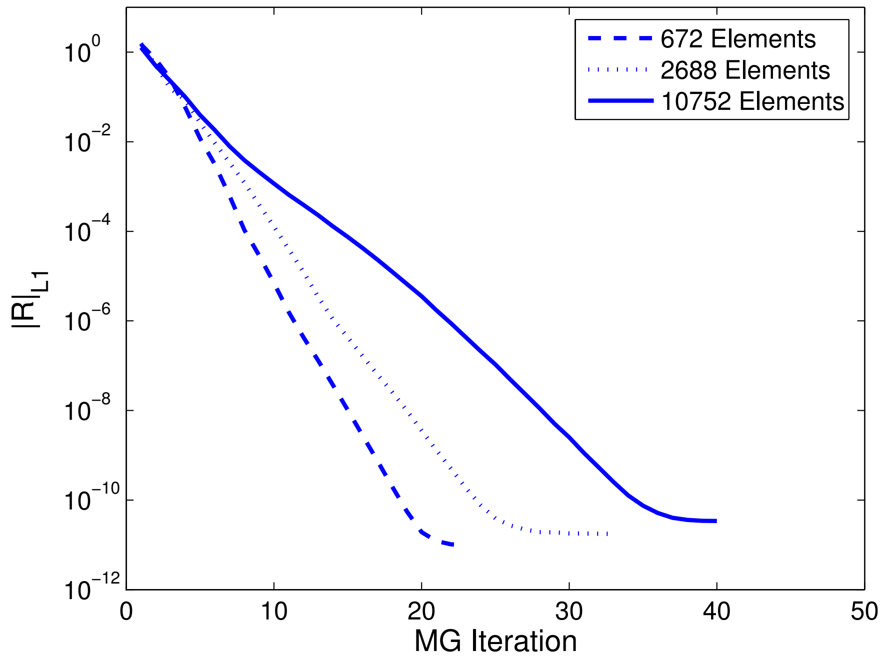
Mach contours

Iterative Behavior: p and h dependence

$p = 1$ convergence



$p = 3$ convergence

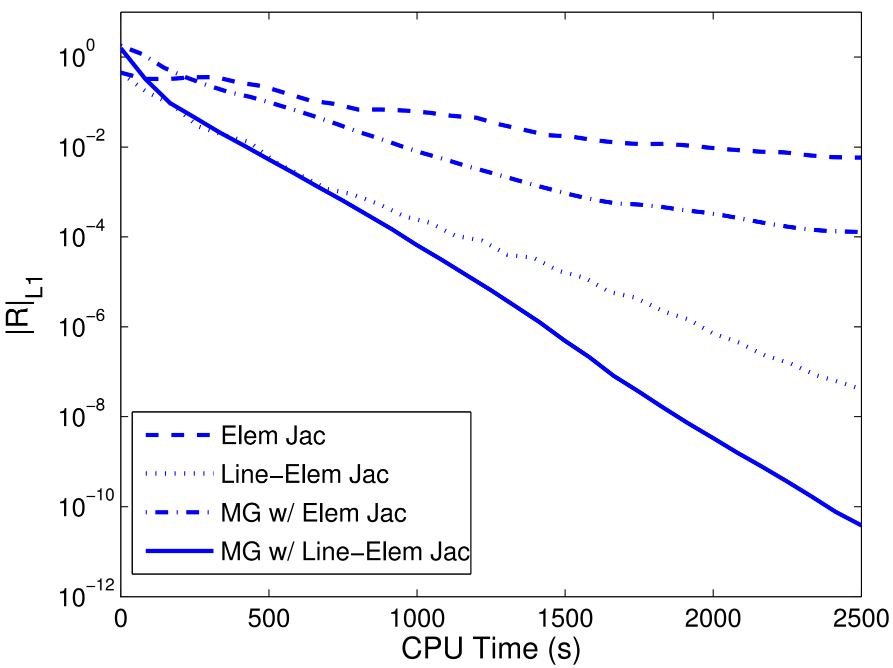


Iterative rate for p-multigrid with line smoothing:

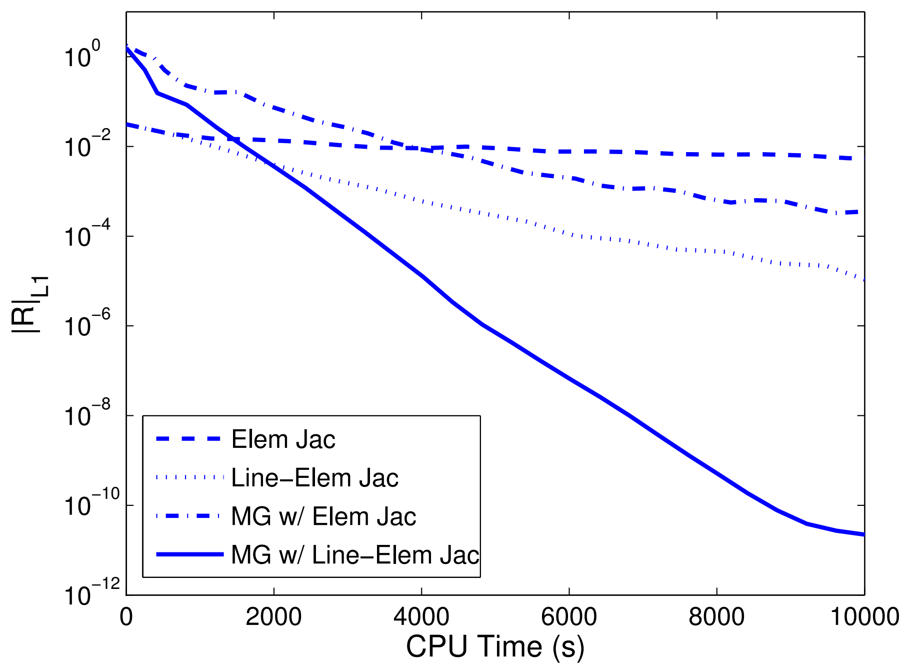
- Nearly p-independent
- Some h-dependence

Comparison of Iterative Algorithms

$p = 1$ convergence

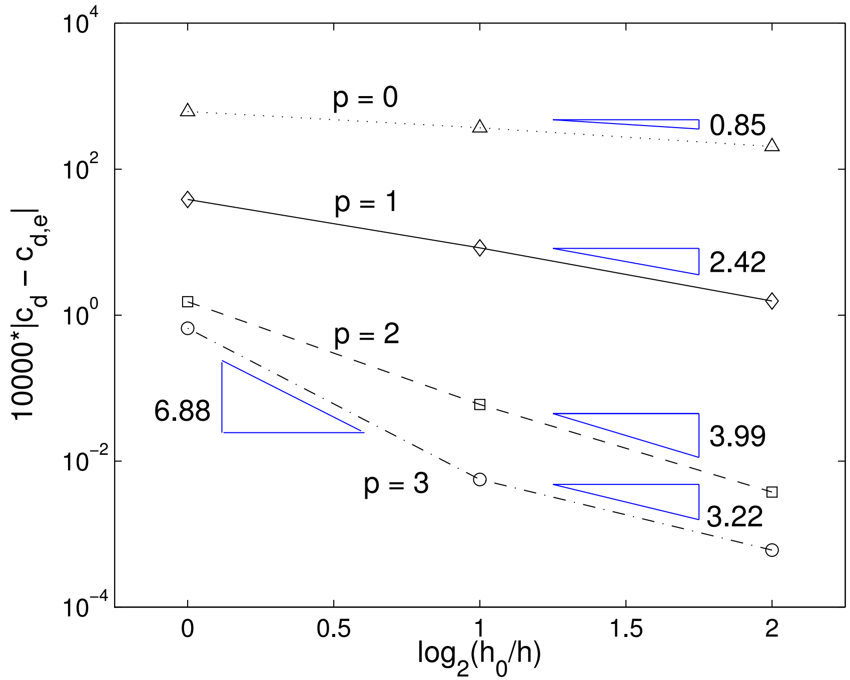


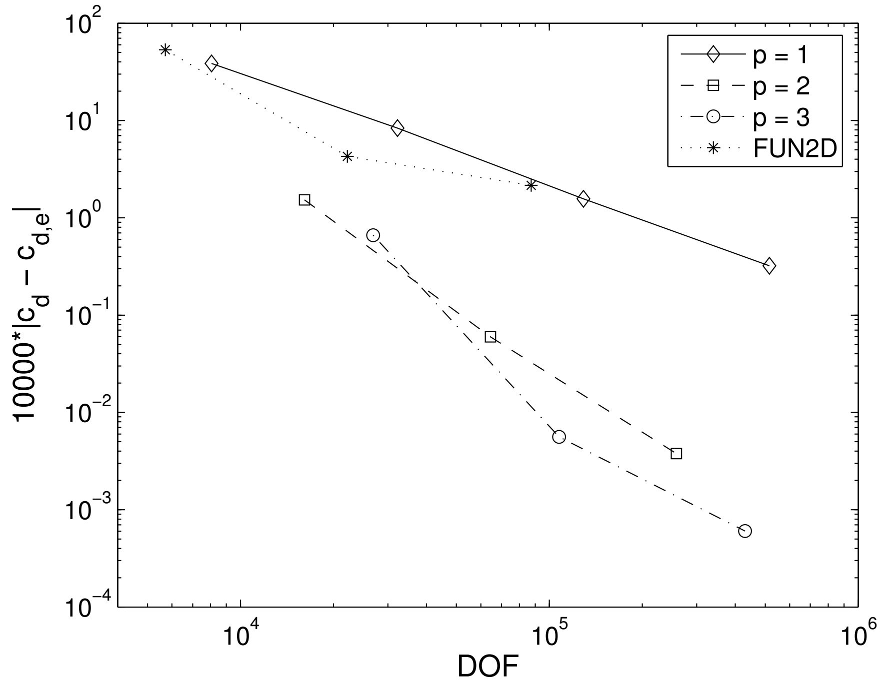
$p = 3$ convergence



p-multigrid with line smoothing increasingly important with higher p

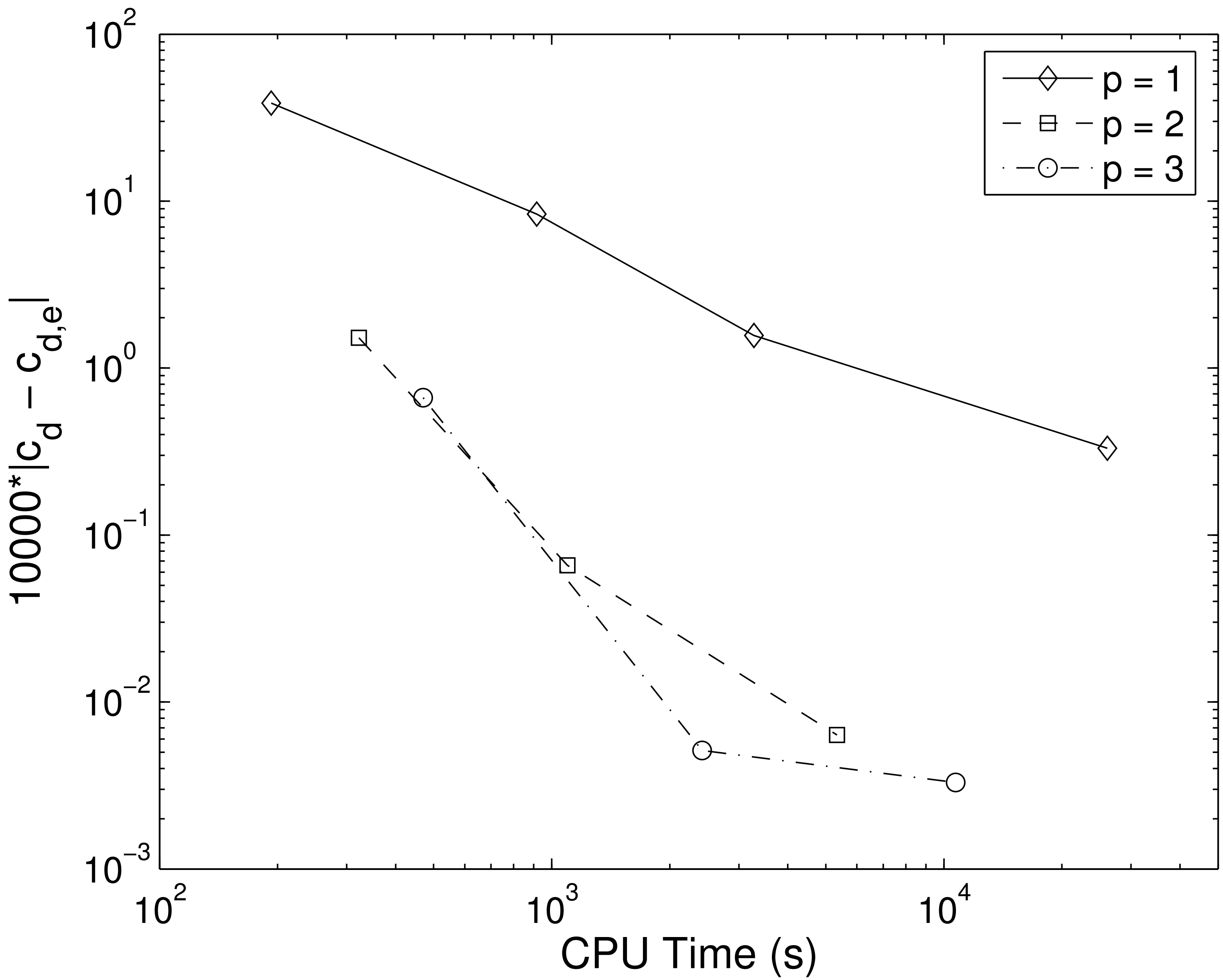
Drag Error Convergence





Note: FUN2D is an unstructured finite volume algorithm developed at NASA Langley by Anderson

CPU Timing



Future Work

- Turbulence modeling
- Shocks
- Adaptation
- Optimization
- Many others