An Introduction to Discontinuous Galerkin Methods for Compressible Flows

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November 5, 2004

Overview

- Motivation: Why develop another CFD algorithm?
- Finite volume methods for hyperbolic conservation laws
- Discontinuous Galerkin (DG) for hyperbolic conservation laws
- DG for elliptic problems
- p-multigrid for higher-order DG discretizations
- Conclusions and future work

Motivation for higher order

- State of CFD in applied aerodynamics
  - Finite-volume with at best second order accuracy
  - Questions exist whether current discretizations are capable of achieving desired accuracy levels in practical time
- Decrease computational time and gridding requirements by increasing solution order

$$\log T = wd\left(-\frac{1}{p}\log E + \log p\right) - \log F + const$$

- T = time to solution
- p = discretization order
- E = desired error level ($E << 1$)
- w = solution complexity
- d = dimension of problem
- F = computational speed

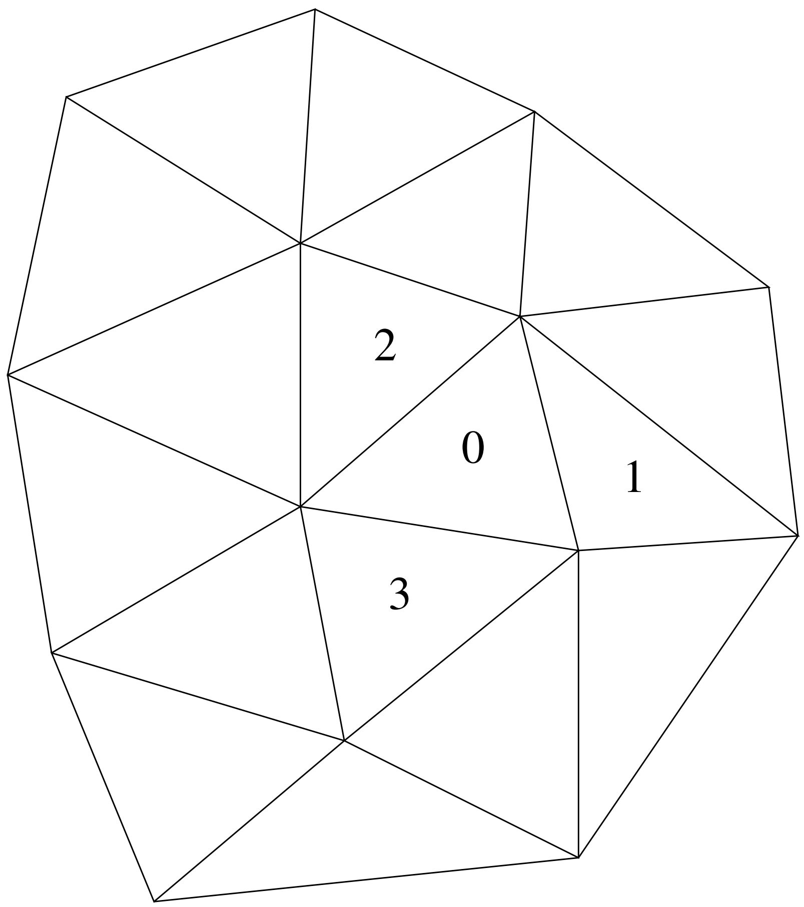
Project-X Goal

- Project X Team Goal:
  - To improve the aerothermal design process for complex 3D configurations by significantly reducing the time from geometry to solution at engineering-required accuracy using high-order adaptive methods

Previous Work

- Extensive work on DG for hyperbolic equations
  - Bassi and Rebay (1997)
  - Cockburn and Shu (1998, 2001)
  - Karniadakis et al. (1998, 1999)
- More recently work begun on elliptic equations
  - Bassi and Rebay (1997,1998)
  - Cockburn and Shu (1998, 2001)
  - Baumann and Oden (1997)
  - Brezzi et al. (1997)
- Only Bassi and Rebay have published RANS results (1997, 2003)

Integral Form of Hyperbolic Conservation Laws



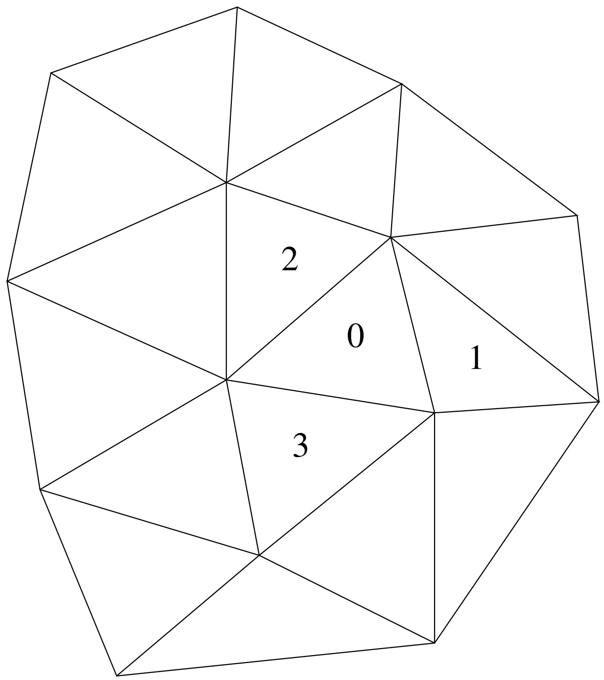
Apply integral conservation law on triangle 0:

$$\frac{d}{dt}\int_{A_0} \mathbf{u}\, d\mathbf{x} + \sum_{k=1}^{3} \int_{0k} \mathcal{F}_i(\mathbf{u}) \cdot \hat{\mathbf{n}}\, ds = 0$$

For Euler equations:

$$\begin{aligned}
\mathbf{u} &= (\rho, \rho u, \rho v, \rho E)^T \\
\mathcal{F}_i &= (\mathbf{F}_i^x, \mathbf{F}_i^y)^T \\
\mathbf{F}_i^x &= \left(\rho u, \rho u^2 + p, \rho u v, \rho u H\right)^T \\
\mathbf{F}_i^y &= \left(\rho v, \rho u v, \rho v^2 + p, \rho v H\right)^T
\end{aligned}$$

First-order Accurate Finite Volume



In each triangle, assume $\mathbf{u}$ is constant.

Apply conservation law on triangle:

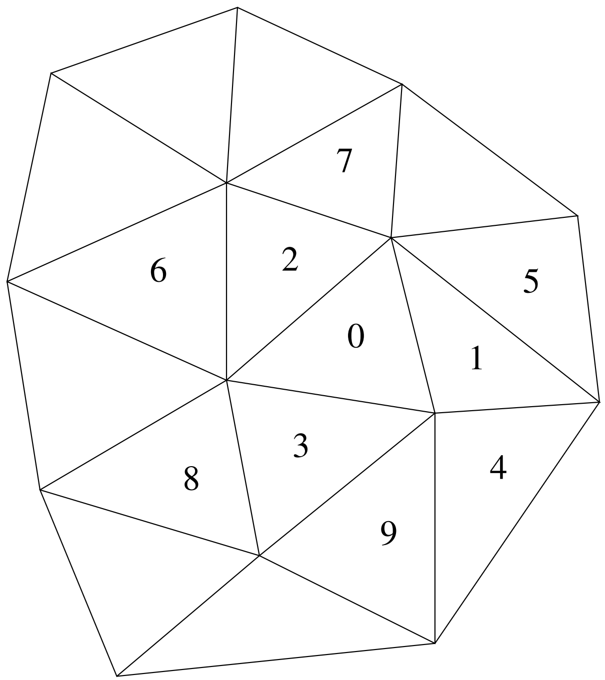
$$\frac{d\mathbf{u}_0}{dt} A_0 + \sum_{k=1}^{3} \int_{0k} \mathcal{H}_i(\mathbf{u}_0, \mathbf{u}_k, \hat{\mathbf{n}}_{0k})\, ds = 0$$

$\mathcal{H}_i(\mathbf{u}_L, \mathbf{u}_R, \hat{\mathbf{n}}_{LR})$ is flux function that determines inviscid flux in $\hat{\mathbf{n}}_{LR}$ direction from left and right states, $\mathbf{u}_L$ and $\mathbf{u}_R$.

Example flux functions: Godunov, Roe, Osher, Van Leer, Lax-Friedrichs, etc.

This discretization has a solution error which is $O(h)$ where h is mesh size.

Second-order Accurate Finite Volume



In each triangle, reconstruct a linear solution, $\tilde{\mathbf{u}}$, using neighboring averages:

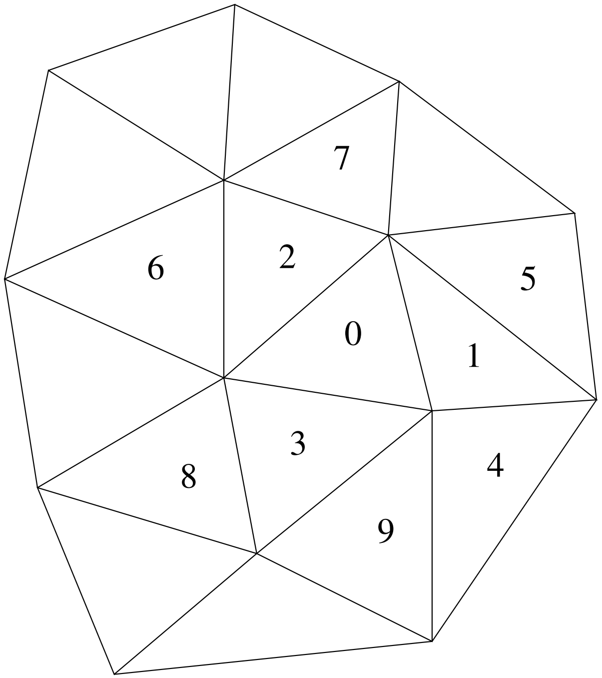
$$\begin{aligned} \tilde{\mathbf{u}}_0 &\equiv \mathbf{u}_0 + (\mathbf{x} - \mathbf{x}_0) \cdot \nabla \mathbf{u}_0, \\ \nabla \mathbf{u}_0 &\equiv \nabla \mathbf{u}_0 \left(\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \right). \end{aligned}$$

Apply conservation law on triangle:

$$\frac{d\mathbf{u}_0}{dt} A_0 + \sum_{k=1}^{3} \int_{0k} \mathcal{H}_i(\tilde{\mathbf{u}}_0, \tilde{\mathbf{u}}_k, \hat{\mathbf{n}}_{0k})\, ds = 0$$

On smooth meshes and flows, solution error is $O(h^2)$.

Pros/Cons of Higher-order Finite Volume



+ Increased accuracy on given mesh without additional degrees of freedom

– Difficulty in achieving higher-order on unstructured meshes and near boundaries

– Stabilizing multi-stage methods necessary for local iterative schemes

– Matrix fill-in increased resulting in high-memory requirements

Instability of Local Iterative Methods

Consider steady state problem and define discrete residual for cell j,

$$\mathbf{R}_j(\mathbf{u}) \equiv \sum_{k=1}^{3} \int_{jk} \mathcal{H}_i(\tilde{\mathbf{u}}_j, \tilde{\mathbf{u}}_k, \hat{\mathbf{n}}_{jk})\, ds = 0.$$

A Jacobi iterative method to solve this problem is,

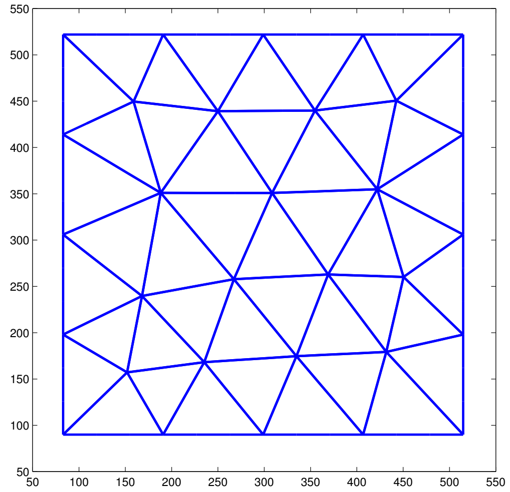
$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \omega \left(\partial \mathbf{R}_j / \partial \mathbf{u}_j\right)^{-1} \mathbf{R}_j(\mathbf{u}).$$

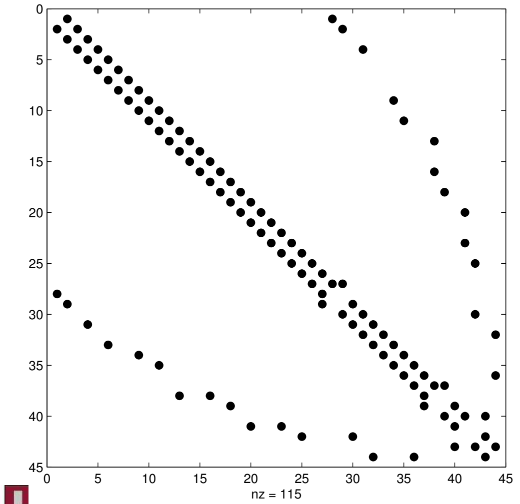
For any finite ω, Jacobi is unstable for higher-order. One solution is a multi-stage method,

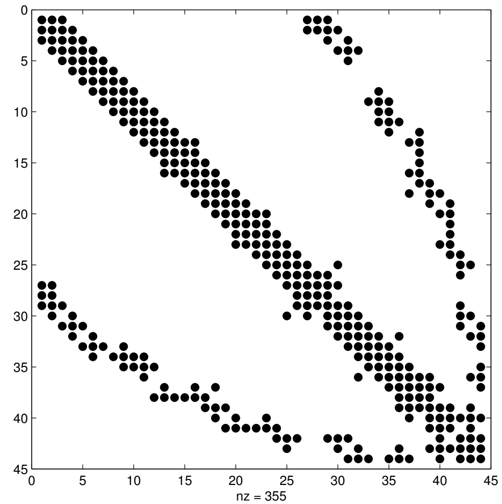
$$\begin{aligned} \hat{\mathbf{u}}_j &= \mathbf{u}_j^n - \hat{\omega} \left(\partial \mathbf{R}_j / \partial \mathbf{u}_j\right)^{-1} \mathbf{R}_j(\mathbf{u}^n) \\ \mathbf{u}_j^{n+1} &= \mathbf{u}_j^n - \omega \left(\partial \mathbf{R}_j / \partial \mathbf{u}_j\right)^{-1} \mathbf{R}_j(\hat{\mathbf{u}}) \end{aligned}$$

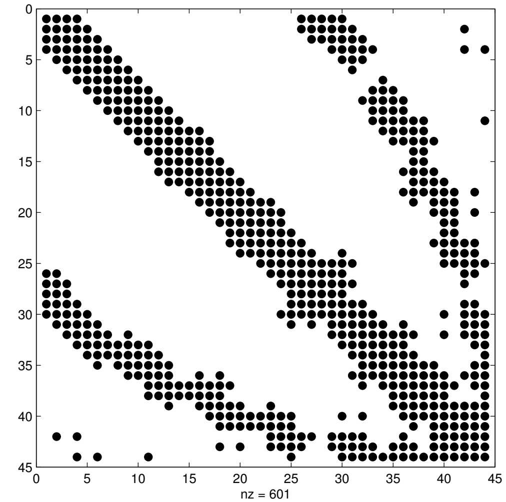
$\Leftarrow$ Requires two residual evaluations.

Matrix Fill for Higher-order Finite Volume









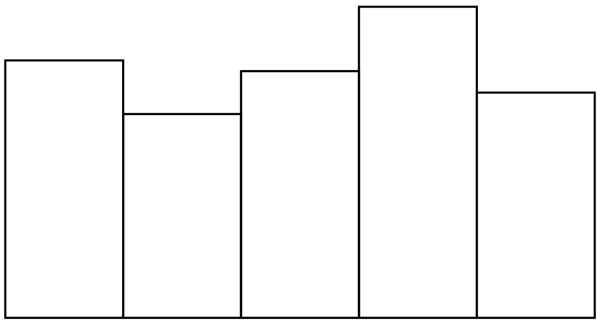
Discontinuous Polynomial Basis

- Triangulate domain Ω into non-overlapping elements $\kappa \in T_h$
- Define function space: Element-wise discontinuous polynomials of degree p

$$\mathcal{V}_h^p = \{\mathbf{v} \in L^2(\Omega) : \mathbf{v}|_\kappa \in P^p(\kappa) : \forall \kappa \in T_h\}$$

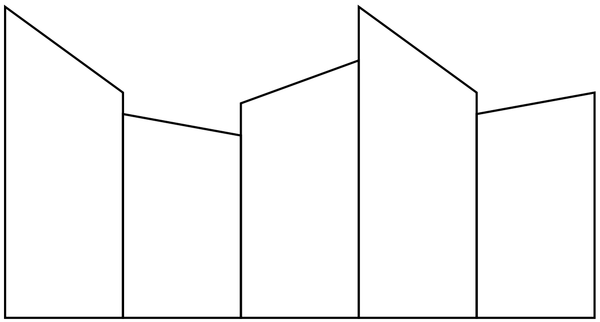
Example of One-Dimensional Bases

$p = 0$ basis



1 DOF/element

$p = 1$ basis



2 DOF/element

DG for Hyperbolic Conservation Laws: Derivation

- Start from strong form of governing equations:

$$\mathbf{u}_t + \nabla \cdot \mathcal{F}_i(\mathbf{u}) = 0.$$

- Look for a solution $\mathbf{u}_h \in \mathcal{V}_h^p$.
- Multiply governing equation by weight function $\mathbf{v}_h \in \mathcal{V}_h^p$ and integrate over element $\kappa \in T_h$:

$$\int_\kappa \mathbf{v}_h^T \left[(\mathbf{u}_h)_t + \nabla \cdot \mathcal{F}_i \right] \, d\mathbf{x} = 0.$$

- Integrate second term by parts (assume interior element):

$$\int_\kappa \mathbf{v}_h^T (\mathbf{u}_h)_t \, d\mathbf{x} - \int_\kappa \nabla \mathbf{v}_h^T \cdot \mathcal{F}_i \, d\mathbf{x} + \int_{\partial\kappa} \mathbf{v}_h^{+T} \mathcal{H}_i(\mathbf{u}_h^+, \mathbf{u}_h^-, \hat{\mathbf{n}}) ds = 0.$$

Relationship of DG to other methods

- Recall DG weighted residual (Reed & Hill, 1973):

$$\int_\kappa \mathbf{v}_h^T (\mathbf{u}_h)_t \, d\mathbf{x} - \int_\kappa \nabla \mathbf{v}_h^T \cdot \mathcal{F}_i \, d\mathbf{x} + \int_{\partial\kappa} \mathbf{v}_h^{+T} \mathcal{H}_i(\mathbf{u}_h^+, \mathbf{u}_h^-, \hat{\mathbf{n}}) ds = 0.$$

- For $p = 0$ solution, this reduces to:

$$(\mathbf{u}_\kappa)_t A_\kappa + \int_{\partial\kappa} \mathcal{H}_i(\mathbf{u}_h^+, \mathbf{u}_h^-, \hat{\mathbf{n}}) ds = 0.$$

- Thus, $p = 0$ DG is identical to first-order finite volume.
- For $p > 0$, DG can be intrepreted as a moment method.
- Moment methods for hyperbolic problems were first suggested by Van Leer (1977) and then developed for the Euler equations by Allmaras (1987, 1989) and later Holt (1992).

DG discretization: Global view

- Find $\mathbf{u}_h \in \mathcal{V}_h^p$ such that $\forall \mathbf{v}_h \in \mathcal{V}_h^p$,

$$\sum_{\kappa \in T_h} \left\{ \int_\kappa \mathbf{v}_h^T (\mathbf{u}_h)_t \, d\mathbf{x} - \int_\kappa \nabla \mathbf{v}_h^T \cdot \mathcal{F}_i \, d\mathbf{x} \right\}$$

$$+ \int_{\Gamma_i} \mathbf{v}_h^{+T} \mathcal{H}_i(\mathbf{u}_h^+, \mathbf{u}_h^-, \hat{\mathbf{n}}) \, ds + \int_{\partial\Omega} \mathbf{v}_h^{+T} \mathcal{H}_i^b(\mathbf{u}_h^+, \mathbf{u}_h^b, \hat{\mathbf{n}}) \, ds = 0.$$

- Boundary conditions enforced weakly through $\mathcal{H}_i^b(\mathbf{u}_h^+, \mathbf{u}_h^b, \hat{\mathbf{n}})$ where $\mathbf{u}_h^b$ is determined from desired boundary conditions and outgoing characteristics.
- For smooth problems, the error of this scheme is expected to be $O(h^{p+1})$.

Pros/Cons of Higher-order DG

– Increased accuracy on given mesh requires additional degrees of freedom

+ Higher-order accuracy not hampered on unstructured meshes nor near boundaries

+ Local iterative methods are stable

+ Matrix fill-in maintains block sparsity of $p = 0$

$$\int_{\kappa} \mathbf{v}_h^T (\mathbf{u}_h)_t \, d\mathbf{x} - \int_{\kappa} \nabla \mathbf{v}_h^T \cdot \mathcal{F}_i \, d\mathbf{x} + \int_{\partial\kappa} \mathbf{v}_h^{+T} \mathcal{H}_i(\mathbf{u}_h^+, \mathbf{u}_h^-, \hat{\mathbf{n}}) ds = 0.$$

Stability of Local Iterative Methods for DG

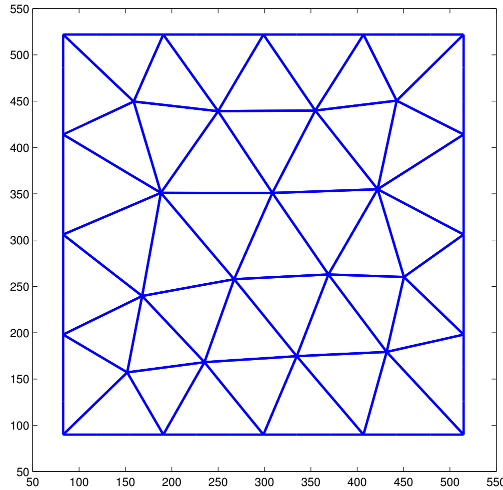
An elemental block Jacobi iterative method to solve this problem is,

$$\mathbf{u}_j^{n+1} = \mathbf{u}_j^n - \omega\left(\partial \mathbf{R}_j / \partial \mathbf{u}_j\right)^{-1} \mathbf{R}_j(\mathbf{u}).$$

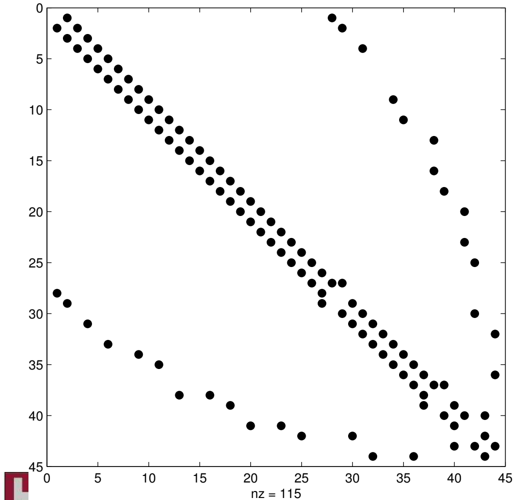
where $\partial \mathbf{R}_j / \partial \mathbf{u}_j$ is the diagonal block for the element j.

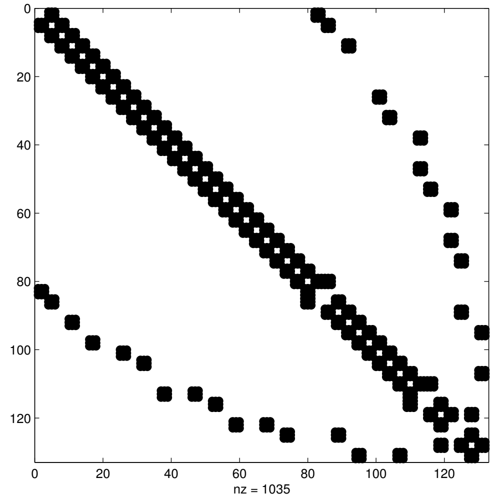
For $0 < \omega < 1$, elemental block Jacobi is stable independent of p.

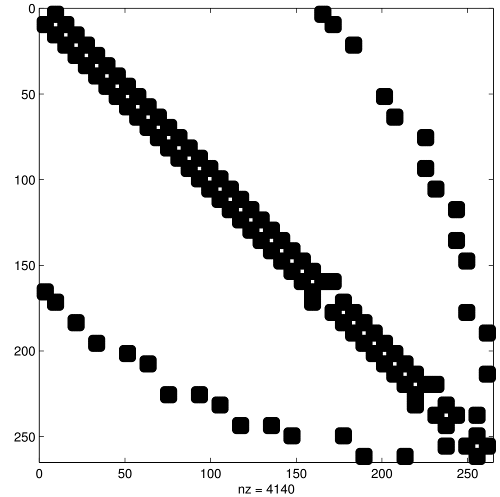
Matrix Fill for Higher-order DG



First-order ($p = 0$) Second-order ($p = 1$) Third-order ($p = 2$)







Navier-Stokes Equations

- Navier-Stokes Equations: $\mathbf{u}_t + \nabla \cdot \mathcal{F}_i(\mathbf{u}) - \nabla \cdot \mathcal{F}_v(\mathbf{u}, \nabla \mathbf{u}) = 0$
- $\mathcal{F}_v = \mathcal{A}_v \nabla \mathbf{u} = (\mathbf{F}_v^x, \mathbf{F}_v^y)$ is the viscous flux vector

$$\mathbf{F}_v^x = \begin{pmatrix} 0 \\ \frac{2}{3}\mu(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) \\ \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ \frac{2}{3}\mu(2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})u + \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})v + \kappa\frac{\partial T}{\partial x} \end{pmatrix},$$

$$\mathbf{F}_v^y = \begin{pmatrix} 0 \\ \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ \frac{2}{3}\mu(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}) \\ \frac{2}{3}\mu(2\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x})v + \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})u + \kappa\frac{\partial T}{\partial y} \end{pmatrix}$$

DG for Elliptic Operators: First Attempt

- Model problem for viscous terms of N-S: 1-D, scalar Poisson's equation

$$-u_{xx} = f \quad \text{on} \quad [-1, 1]$$

- Proceed as for Euler:
  - Triangulate domain into non-overlapping elements $\kappa \in T_h$
  - Define solution and test function space $\mathcal{V}_h^p$
- Discrete formulation: Find $u_h \in \mathcal{V}_h^p$ such that $\forall v_h \in \mathcal{V}_h^p$,

$$\sum_{\kappa \in T_h} \left\{ - \left[v_h \widehat{u_x} \right]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} + \int_\kappa (v_h)_x (u_h)_x dx \right\} = \sum_{\kappa \in T_h} \left\{ \int_\kappa v_h f dx \right\}$$

- Need to define $\widehat{u_x}$

DG for Elliptic Operators: First Attempt

- No upwinding mechanism $\Rightarrow$ choose central flux

$$\widehat{u_x} = \frac{1}{2}((u_h)_x^L + (u_h)_x^R)$$

- Discrete formulation becomes: Find $u_h \in \mathcal{V}_h^p$ such that $\forall v_h \in \mathcal{V}_h^p$,

$$\sum_{\kappa \in T_h} \Big\{ - \Big[\frac{1}{2} v_h((u_h)_x^L + (u_h)_x^R)\Big]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} + \int_\kappa (v_h)_x (u_h)_x dx \Big\} = \sum_{\kappa \in T_h} \Big\{ \int_\kappa v_h f dx \Big\}$$

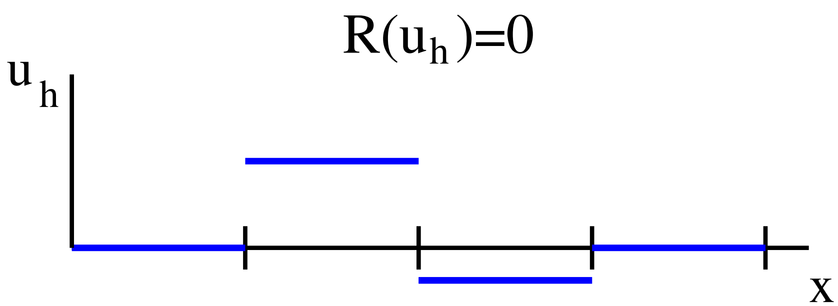
- PROBLEM: Scheme is inconsistent!

Inconsistency

- Examine Laplace's equation with homogeneous Dirichlet BCs

$$
\begin{aligned}
-u_{xx} &= 0 \quad \text{on} \quad [-1,1] \\
u(-1) = u(1) &= 0
\end{aligned}
$$

- Exact solution: $u(x) = 0$



- If $(u_h)_x = 0$ everywhere, discrete equations satisfied exactly regardless of magnitude of u_h

First Order System Approach

- Introduce new variable, $q = u_x$, such that

$$\begin{aligned} -q_x &= f \\ q - u_x &= 0 \end{aligned}$$

- Discrete formulation: Find $u_h \in \mathcal{V}_h^p$ and $q_h \in \mathcal{V}_h^p$ such that $\forall v_h \in \mathcal{V}_h^p$ and $\forall \tau_h \in \mathcal{V}_h^p$,

$$\begin{aligned} \sum_{\kappa \in T_h} \left\{ -\left[v_h \widehat{q} \right]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} + \int_\kappa (v_h)_x q_h dx \right\} - \sum_{\kappa \in T_h} \left\{ \int_\kappa v_h f dx \right\} &= 0 \\ \sum_{\kappa \in T_h} \left\{ \int_\kappa \tau_h q_h dx + \int_\kappa (\tau_h)_x u_h dx - \left[\tau_h \widehat{u} \right]_{x_{\kappa-1/2}}^{x_{\kappa+1/2}} \right\} &= 0 \end{aligned}$$

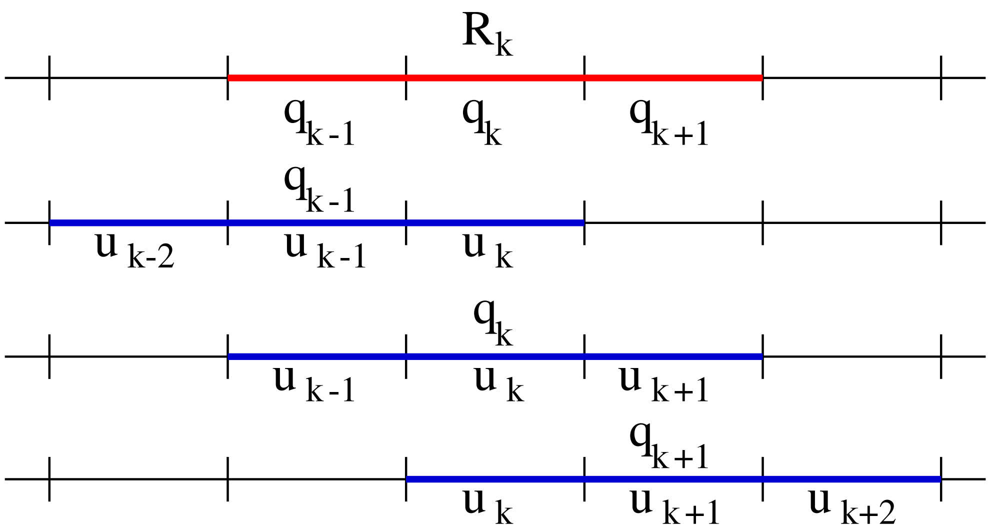
- Need to choose $\widehat{q}$ and $\widehat{u}$

BR1 Scheme

- No upwinding mechanism $\Rightarrow$ choose central fluxes

$$\widehat{u} = \frac{1}{2}(u_h^L + u_h^R); \quad \widehat{q} = \frac{1}{2}(q_h^L + q_h^R)$$

- Sub-optimal order of accuracy for odd p
- Stencil no longer compact



BR1 Scheme

- Define jump, $[\![\cdot]\!]$, and average, $\{\cdot\}$, operators:

$$[\![s]\!] = s^L - s^R \quad \text{and} \quad \{s\} = \frac{1}{2}(s^L + s^R)$$

- Central fluxes become

$$\widehat{u} = \{u_h\}; \quad \widehat{q} = \{(u_h)_x\} - \{\delta\}$$

- δ given by following problem: Find $\delta \in \mathcal{V}_h^p$ such that $\forall \tau_h \in \mathcal{V}_h^p$,

$$\sum_{\kappa \in T_h} \int_\kappa \tau_h \delta dx = \sum_n \left[[\![u_h]\!] \{\tau_h\} \right]$$

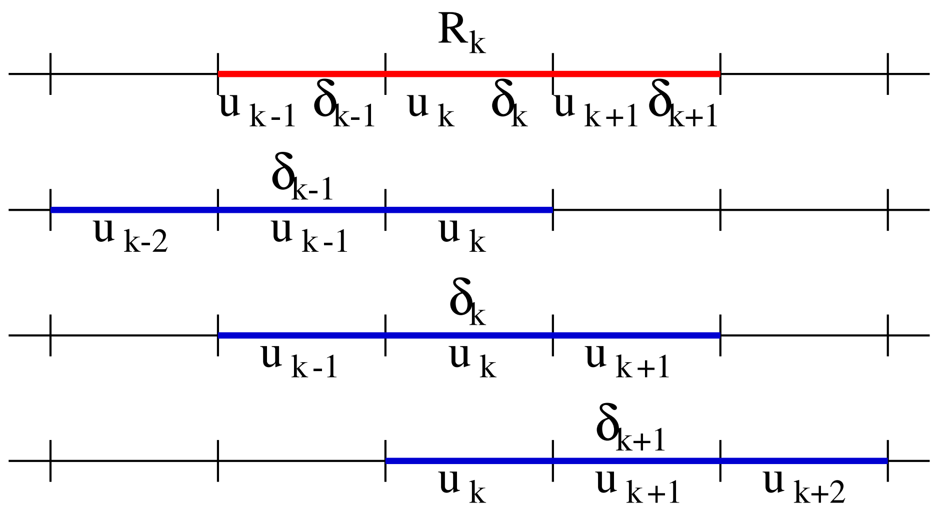
BR1 Scheme

- BR1 becomes: Find $u_h \in \mathcal{V}_h^p$ and such that $\forall v_h \in \mathcal{V}_h^p$,

$$\sum_{\kappa \in T_h} \int_\kappa (v_h)_x (u_h)_x dx$$

$$-\sum_n \left[[\![u_h]\!]\{(v_h)_x\} + [\![v_h]\!](\{(u_h)_x\} - \{\delta\}) \right] = \sum_{\kappa \in T_h} \int_\kappa v_h f dx$$

- Stencil extended by δ dependence on u_h



BR2 Scheme

- Goal: Eliminate extended stencil
- Approach: Modify auxiliary variable, δ, previously defined by:

$$\sum_{\kappa \in T_h} \int_{\kappa} \tau_h \delta dx = \sum_{n} \left[[\![u_h]\!] \{\tau_h\} \right]$$

- New variable, δ_f, given by: Find $\delta_f \in \mathcal{V}_h^p$ such that $\forall \tau_h \in \mathcal{V}_h^p$,

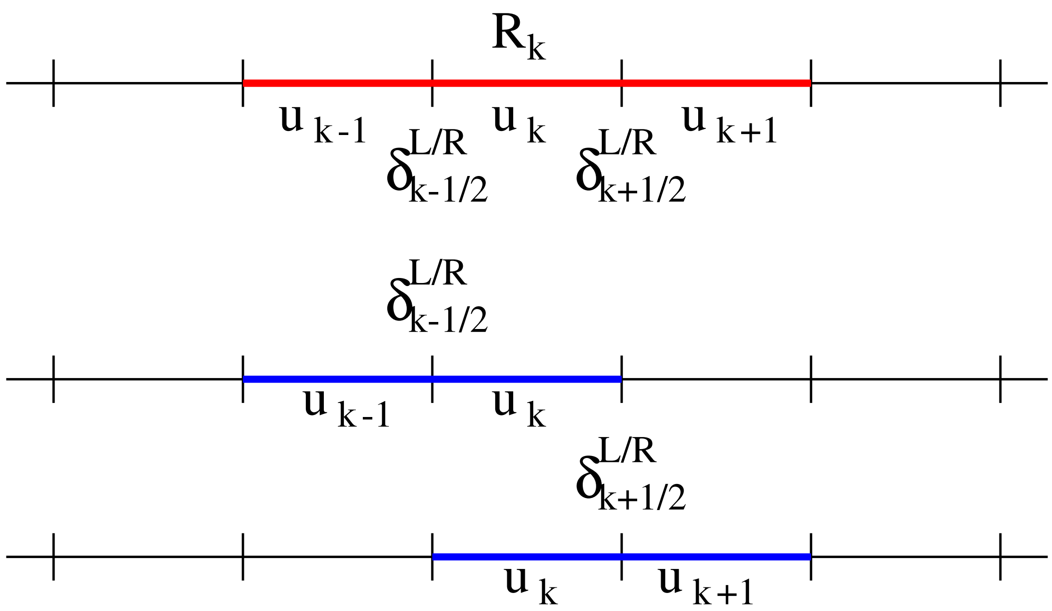
$$\int_{\kappa^{L/R}} \tau_h \delta_f^{L/R} dx = \left[[\![u_h]\!] \{\tau_h\}^{L/R} \right]_{n_f}$$

- New fluxes have same form as before

$$\widehat{u} = \{u_h\}; \quad \widehat{q} = \{(u_h)_x\} - \eta_f \{\delta_f\}$$

BR2 Scheme

- Replacing $\{\delta\}$ in BR1 by $\eta_f\{\delta_f\}$ gives BR2
- For proper choice of η_f, can prove optimal order of accuracy
- Stencil is compact



Iterative Solver

- Use work by Fidkowski and Darmofal (2004) on solution of DG discretization of Euler equations
- Nonlinear discrete equations can be written

$$\mathbf{R}(\mathbf{u}_h) = 0$$

- Use a preconditioned iterative scheme

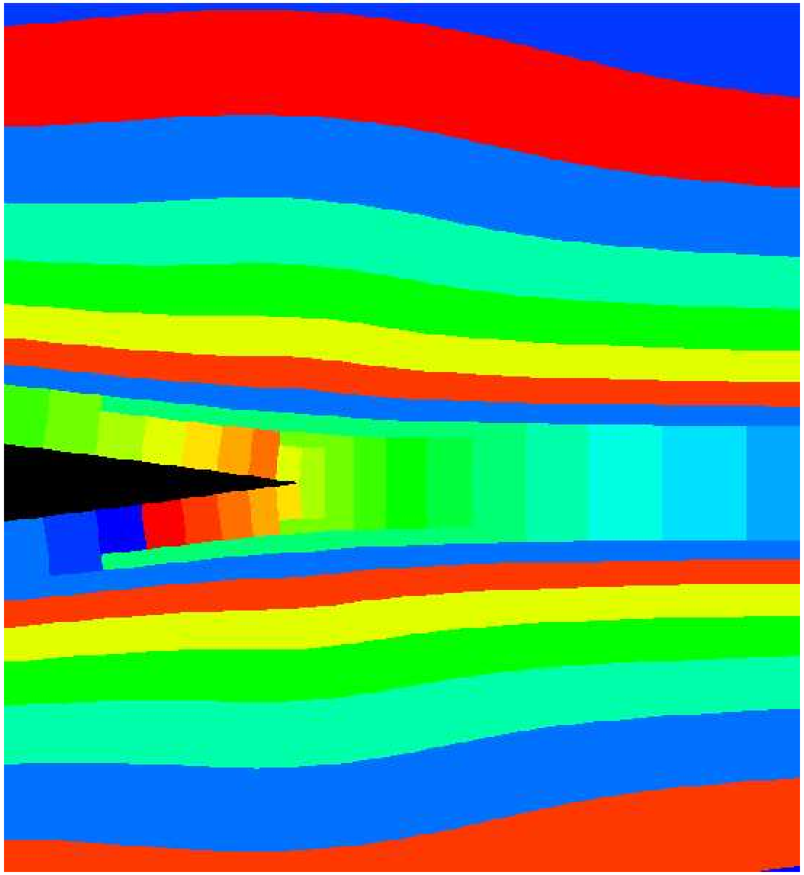
$$\mathbf{u}_h^{n+1} = \mathbf{u}_h^n - \mathbf{P}^{-1}\mathbf{R}(\mathbf{u}_h^n)$$

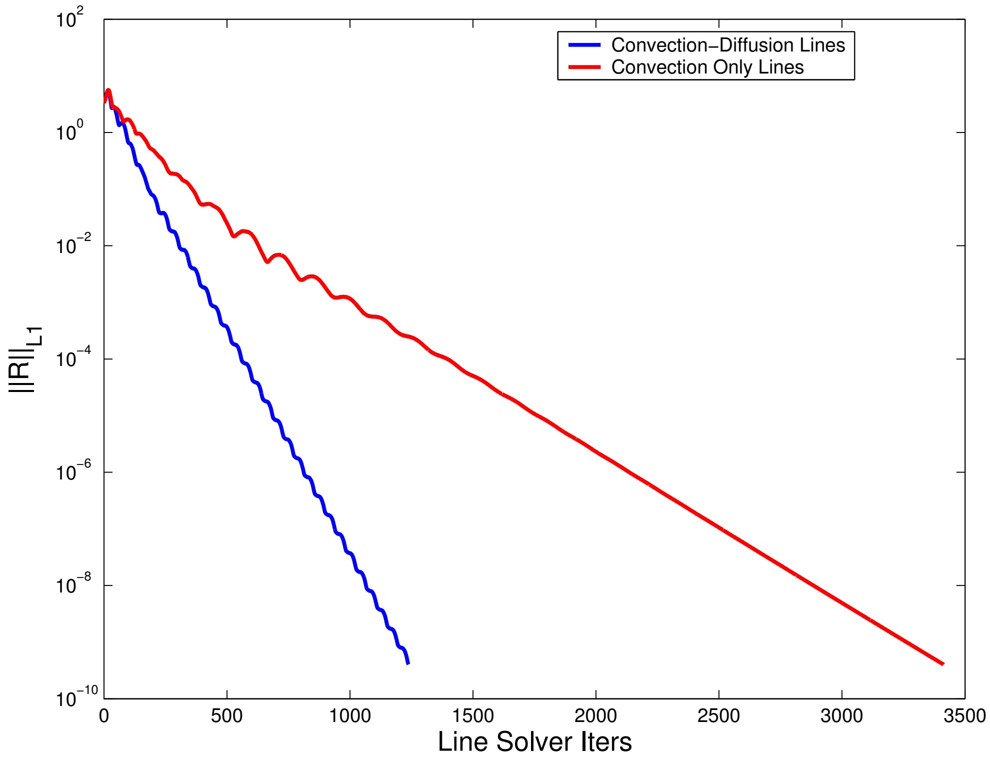
- Preconditioner
  - Block-element smoothing
    - $\mathbf{P} = \mathbf{M}_{block} \Rightarrow$ Block diagonal of the Jacobian
  - Line-element smoothing
    - $\mathbf{P} = \mathbf{M}_{line} \Rightarrow$ Block tridiagonal systems from Jacobian

Line Solver

- Motivation: Transport of information in Navier-Stokes equations characterized by convection-diffusion like phenomena
  - Inviscid regions: Information follows characteristic directions set by convection
  - Viscous regions: Diffusion effects can be as strong or stronger than convection
- Procedure:
  - Construct lines of elements based on measure of influence
  - Build and invert $\mathbf{M}_{line}$, which is a set of block tridiagonal systems from the full Jacobian

Example Lines and Performance



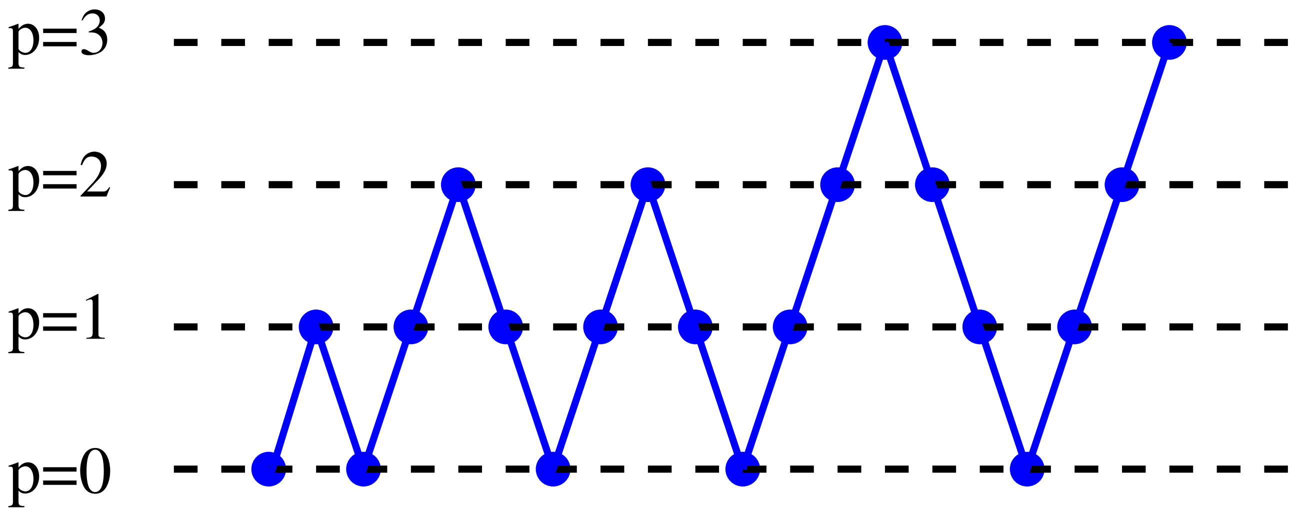


p-Multigrid: Motivation

- Observation: Smoothers are inefficient at eliminating low frequency error modes on fine level
- h-Multigrid
  - Spatially coarse grid used to correct solution on fine grid
  - Grid coarsening is complex on unstructured meshes
- p-Multigrid (Ronquist & Patera, Helenbrook et al., Fidkowski & Darmofal)
  - Low order $(p-1)$ approximation used to correct high order (p) solution
  - Natural implementation in DG FEM discretization on unstructured meshes

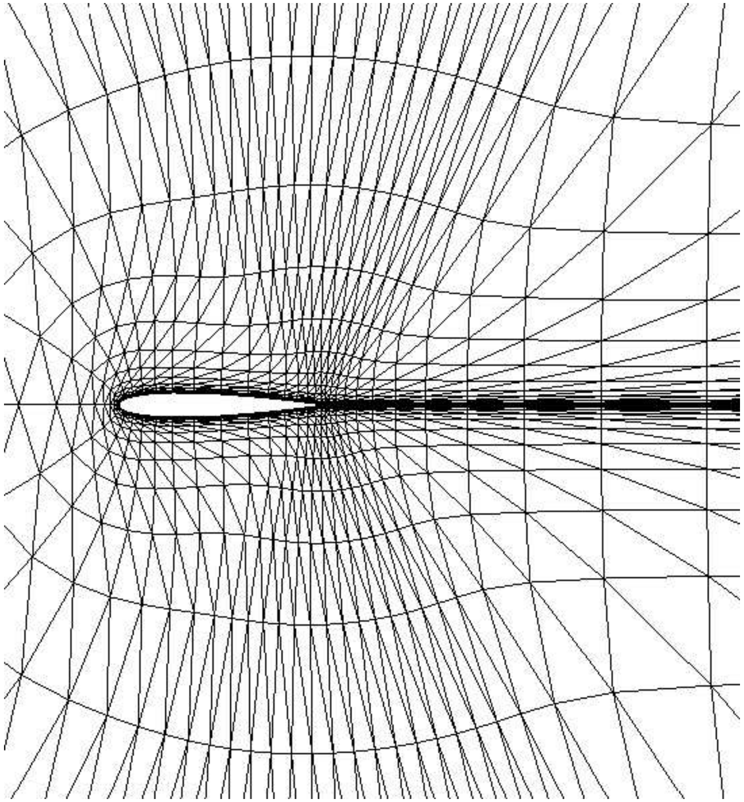
p-Multigrid: Full Multigrid

- Full Approximation Scheme (FAS) used
- Line solver used as smoother

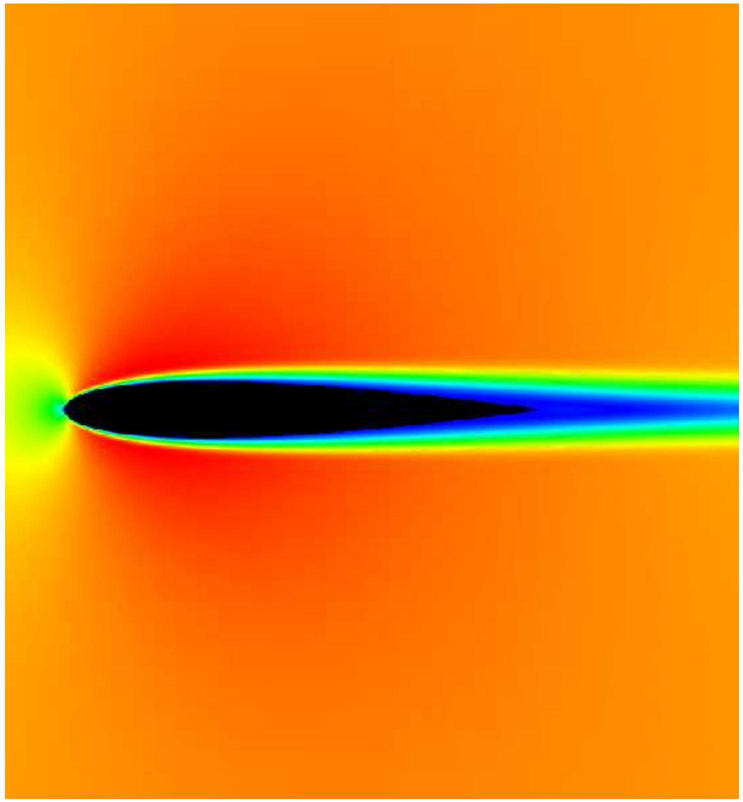


NACA 0012 Test Case

$M = 0.5$, $Re = 5000$, $\alpha = 0$
Grids are from Swanson at NASA Langley

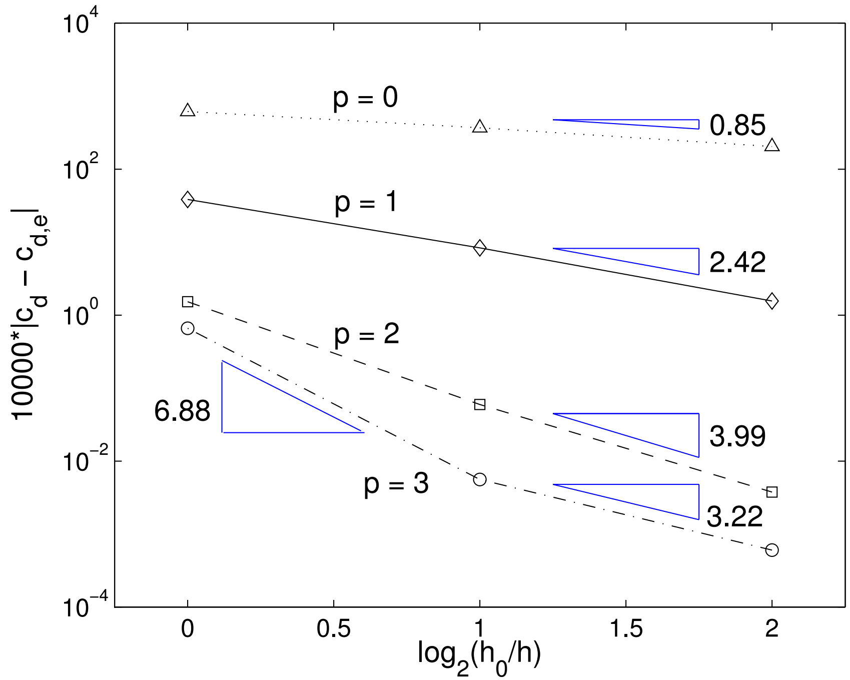


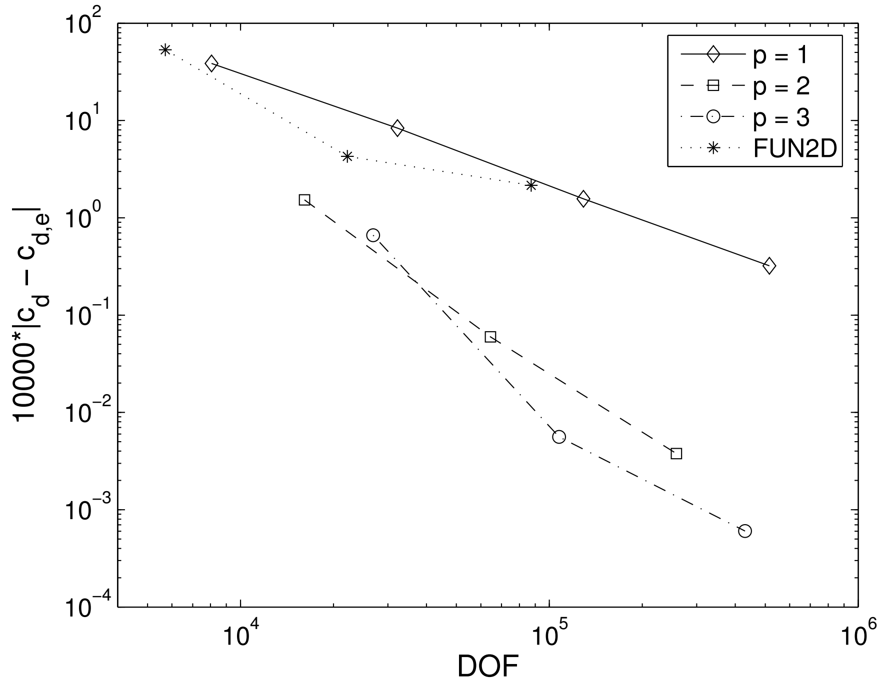
2112 element mesh



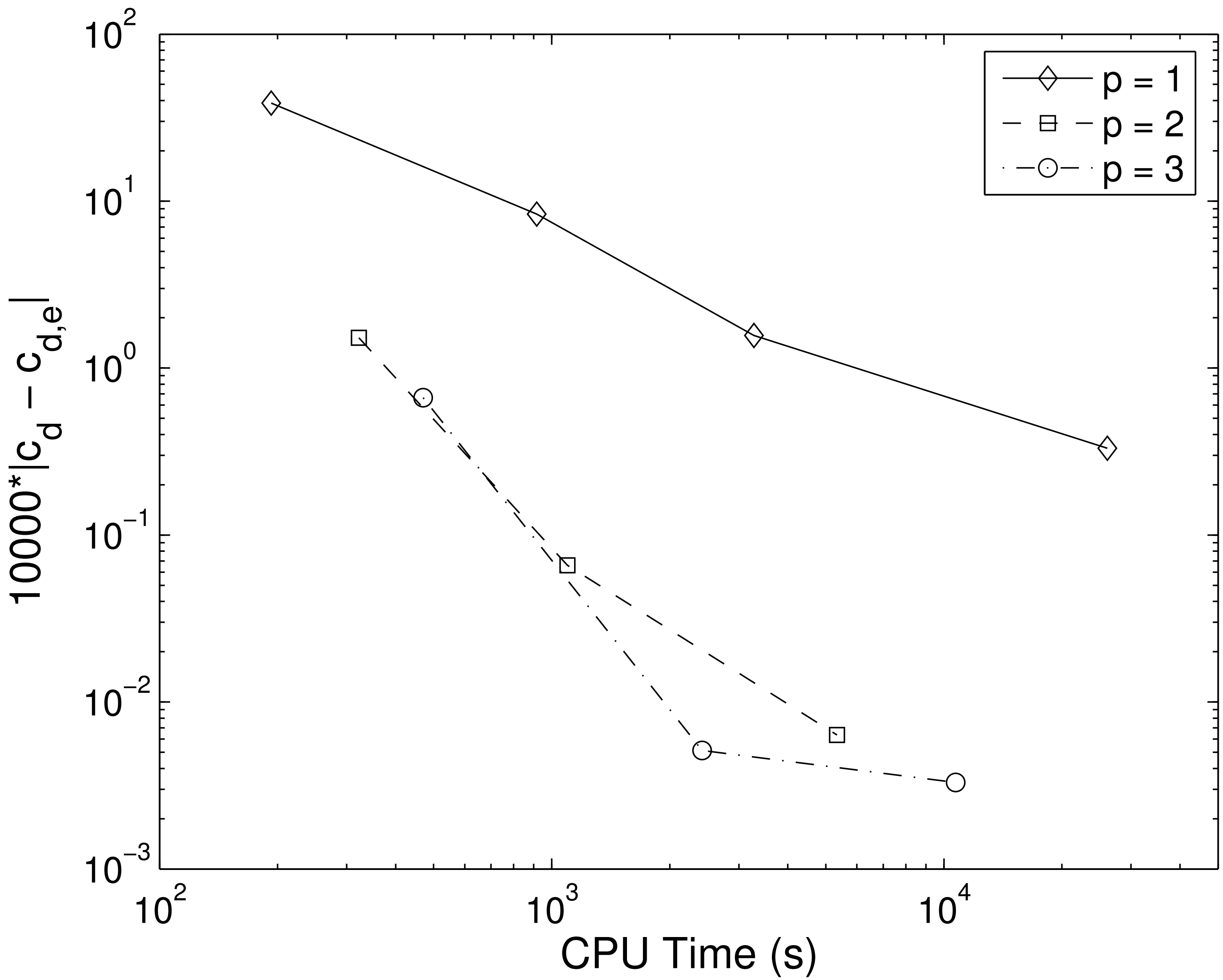
Mach contours

Drag Error Convergence





CPU Timing



p = 1
p = 2
p = 3
$10000*|c_d - c_{d,e}|$
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{2}
10^{3}
10^{4}
CPU Time (s)

Future Work

- Turbulence modeling (Todd)
- Shocks (Jean-Baptiste & Garrett)
- Adaptation (Chris & Mike)
- Optimization (James)
- Many others

Thanks to the entire Project-X crew... this is their work!