16.06 Supplementary Math Notes

Doug Allaire and Karen Willcox

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Contents

1 Introduction

1.1 Purpose and Objectives

This document is intended to help you with the math skills that you will need for 16.06. The idea is to create an explicit linking of the mathematics courses that you took in freshman and sophomore year and the skills that you will need for 16.06. This document will provide you with a comprehensive list of mathematics resources should you need to do further review.

Note that many of the engineering concepts and skills that we will be learning in 16.06 depend directly on the math you learned in 18.01, 18.02 and 18.03. In most cases, we will not be spending lecture time to review these math skills in class. It is therefore very important that you feel comfortable with the math so that you can focus on achieving the 16.06 learning objectives.

1.2 Document Overview

This document is organized in a lecture-by-lecture format that reintroduces mathematical tools as they are used in class. Each section contains some math notes for selected topics as well as a list of references of where each concept was taught in the introductory math courses. Each set of notes will be given out before the corresponding lecture and you should review them before class. For each lecture, you will see a list of the math topics that arise in that lecture. For each topic, there is a list of the specific skills that you will require. Some specific examples are given and a list of resources is provided. If you do not feel comfortable with any of the skills that are listed, you should go back and review these resources.

If there are specific math skills in with you feel particularly weak, we are willing to provide extra review during recitation sessions. Please communicate any requests to the faculty or teaching assistants.

For a complete listing of important math skills and how they relate to Aero/Astro core classes, see Table 2 in http://web.mit.edu/kwillcox/www/WillcoxASEE04.pdf

1.3 Acknowledgments

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2 Lecture 2: Introduction to Feedback Control

Lecture 2 Math Topics

- Functions
- Linearization
- First-Order Ordinary Differential Equations
- Laplace Transforms

2.1 Functions

Required Skills

- Understand the concept of an independent and dependent variables (argument of a function)

"A *function* is a rule that defines how the elements of one set are transformed into the elements of another set (a *set* is a collection of a finite or infinite number of elements). For example, let us consider the set containing all people in the United States and the set of all positive integers with 9 or fewer digits. Then we can think about the Social Security Number system as providing a function from the first set to the second, because for every person we can define the value of our function to be his/her Social Security Number. The first set, the set containing all people in the United States, is called the *domain* of the function." [aa-math website]

"Another example is the function $\sin(x)$. It defines a transformation of a real number to another real number. It is well known that $\sin(x)$ never results in values with magnitude larger than 1, so actually a 'smaller' set can be used for the possible values of the function. The 'smallest' set we can choose in this case is the interval [-1,+1], which is called the *range* of the function." [aa-math website]

Using a different set with the same rule is something you will encounter often in control theory and it will usually take the form of $f(t)$ and $f(t-\tau)$. If you feel uncomfortable with this concept, then you need to do some review!

For more on functions please refer to

Simmons, George F. *Calculus with Analytic Geometry* 2nd ed. pages 22-37. [?]

In the supplementary math notes

The functions you will encounter in the math notes for lecture 2 include $f(x)$, $f(0)$, $f(x_0)$, $f(x_0+\Delta x)$, $\sin\theta$, $\cos\theta$, $P(x)$, $Q(x)$, $y(x)$, and $\Omega(t)$.

In the lecture notes

The functions that you will come across in the lecture 2 notes include $r(t)$, $c(t)$, $R(s)$, $G(s)$, and $C(s)$. You should recognize the functions involving a t as functions of the time domain and the functions involving an s as functions of the Laplace domain.

2.2 Linearization

Required Skills

- Linearize first- and second-order time-varying systems about some operating point

Taylor Series Expansion

Linearization is dealt with in much greater detail at http://web.mit.edu/aa-math/www/. In these notes we will only look at the Taylor series and in particular at the Taylor series expansion of the sine function. Most of what is written here has been taken directly from the aa-math website.

Linearization in general is done with the help of Taylor series. Linearization around a point P means approximating the function in the neighborhood of P. Consider a general function $f(x)$. The Taylor expansion of the function $f(x)$ around $x = 0$ is:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + ... + \frac{f^n(0)}{n!}x^n + ... \tag{1}$$

where f' represents the first derivative of f with respect to x, f'' represents the second derivative of f with respect to x, and so on. The expansion of $f(x)$ around any point x_0 is a generalization of the above:

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}\Delta x^2 + ... + \frac{f^n(x_0)}{n!}\Delta x^n + ... \tag{2}$$

Note that the function and all its derivatives are evaluated at the linearization point x_0.

To linearize the expressions in equations (1) and (2), we drop off the higher order terms as follows:

$$f(x) \approx f(0) + f'(0)x \tag{3}$$
$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x \tag{4}$$

Consider the Taylor series for the $\sin\theta$ around $\theta = 0$. Since the derivatives of the sine function are periodic:
$\cos\theta$, $-\sin\theta$, $-\cos\theta$, $\sin\theta$, and $\cos 0 = 1$, $\sin 0 = 0$, we get for the Taylor series of the sine function around $\theta = 0$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ... + (-1)^k \frac{\theta^{2k+1}}{(k+1)!} \tag{5}$$

To get a linear approximation for the sine function around zero, we see from equation (3) that we must drop the higher order terms. We can do this if $\theta \ll 1$, since $\theta^3, \theta^5, \ldots$ will be much smaller that θ. When we drop the higher order terms for the Taylor series expansion of sine we are left with

$$\sin\theta \approx \theta \tag{6}$$

which is valid only when $\theta << 1$, meaning when θ is small.

Resources

- http://web.mit.edu/aa-math/www/ Discusses linearization and Taylor series expansion
- 18.02, Unit 2, Lecture 8 - *Approximations*, Unit 6 - *Other Topics*, Lecture 32 - *Infinite Series*

- 18.03, Edwards and Penney, Section 3.1, *Introduction and Review of Power Series*, makes the connection between power series and Taylor series
- Other suggested sources
  - http://www-math.mit.edu/18.013A/chapter10/section02.html
  - http://www-math.mit.edu/∼djk/18_01/chapter06/contents.html

Example from Lecture: Pilot roll control of an airplane

Dynamics of the situation are as follows:

- change of heading: requires a horizontal force
- tip lift vector: requires angular acceleration
- create roll moment: ailerons change camber
- move ailerons by displacing control wheel

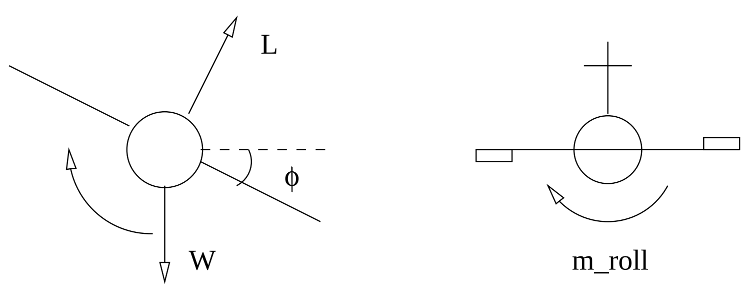

Figure 1: Roll

To derive a **linearized model** to get the appropriate differential equation we must assume small angles for the angle of flap deflection (δ). This is because the moments in this example actually involve $\sin\delta$, which is nonlinear. By using the Taylor series expansion of the sine function as shown above, we see that for small angles $\sin\delta \approx \delta$. Thus, by using δ we can linearize the differential equation, (making it much easier to solve), while still providing a good approximation to the actual system.

Now we may write the equation as:

$$I\ddot{\phi} = \sum M = M_\delta \delta - M_{\dot{\phi}}\dot{\phi} \tag{7}$$

$$I\ddot{\phi} + M_{\dot{\phi}}\dot{\phi} = M_\delta \delta \tag{8}$$

Let $\dot{\phi} = \Omega$, then we have

$$\frac{I}{M_{\dot{\phi}}}\dot{\Omega} + \Omega = \frac{M_\delta}{M_{\dot{\phi}}}\delta \tag{9}$$

Equation (9) is a linear ordinary differential equation that we now would like to solve for Ω.

2.3 Solving a First-Order Linear ODE: Integrating Factors

Required Skills

- Solve linear, constant-coefficient, first–order ODEs

This standard approach to solving linear first order ODEs was taught in 18.03. The material shown here can be found in the 18.03 textbook, *Elementary Differential Equations with Boundary Value Problems* (4th ed.), by Edwards and Penney, in section 1.5. [?] The book discusses the standard technique of solving first-order linear ODEs with the help of an integrating factor as follows.
"If a linear first-order equation takes the form

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{10}$$

Then multiply each side of the equation by an integrating factor of the form

$$e^{\int P(x)dx} \tag{11}$$

The result is

$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = Q(x)e^{\int P(x)dx} \tag{12}$$

Because

$$D_x\left[\int P(x)dx\right] = P(x) \tag{13}$$

the left-hand side is the derivative of the product $y(x)e^{\int P(x)dx}$, thus equation (7) is equivalent to

$$D_x\left[y(x)e^{\int P(x)dx}\right] = Q(x)e^{\int P(x)dx} \tag{14}$$

Integration of both sides yields

$$y(x)e^{\int P(x)dx} = \int (Q(x)e^{\int P(x)dx})dx + C \tag{15}$$

Solving for y, we obtain the general solution

$$y(x) = e^{-\int P(x)dx}\left[\int (Q(x)e^{\int P(x)dx})dx + C\right] \tag{16}$$

Equation (16) should not be memorized. Instead, the method of arriving at equation (16) should be followed when solving this type of equation.

Method: Solution of first-order equations

1. Begin by calculating the integrating factor
2. Then multiply both sides of the differential equation by it
3. Next, recognize that the left-hand side of the resulting equation as the derivative of a product
4. Finally, integrate the equation and solve for y"

Let's now solve our roll model using the standard integrating factor approach. Note that for this equation, Ω = y and x = t.
We begin with

$$\frac{I}{M_{\dot{\phi}}}\dot{\Omega} + \Omega = \frac{M_\delta}{M_{\dot{\phi}}}\delta \tag{17}$$

Dividing both sides of the equation by $\frac{I}{M_{\dot\phi}}$ yields

$$\dot\Omega + \frac{M_{\dot\phi}}{I}\Omega = \frac{M_\delta}{I}\delta \tag{18}$$

Equation (12) is clearly in the form of equation (4) where the P term and the Q term are both constant functions of time. Thus we may use the standard integrating factor approach.
Since $P(t) = \frac{M_{\dot\phi}}{I}$, which is constant, the integrating factor is

$$e^{\int \frac{M_{\dot\phi}}{I}dt} = e^{\frac{M_{\dot\phi}}{I}t} \tag{19}$$

and letting $T = I/M_{\dot\phi}$ we have

$$integratingfactor = e^{t/T} \tag{20}$$

Multiplying both sides by the integrating factor yields

$$e^{t/T}\left[\dot\Omega + \frac{M_{\dot\phi}}{I}\Omega\right] = \left(\frac{M_\delta}{I}\delta\right)e^{t/T} \tag{21}$$

The left-hand side is the derivative of the product $e^{t/T}\Omega$, thus we have

$$D_t\left[e^{t/T}\Omega\right] = \left(\frac{M_\delta}{I}\delta\right)e^{t/T} \tag{22}$$

Integrating both sides of (16) we have

$$e^{t/T}\Omega = \frac{M_\delta}{I}\delta\int e^{t/T}dt \tag{23}$$

The integration yields

$$e^{t/T}\Omega = \frac{M_\delta}{I}\delta\left(Te^{t/T} + C\right) \tag{24}$$

Remembering that $T = \frac{I}{M_{\dot\phi}}$ we have

$$e^{t/T}\Omega = \frac{M_\delta}{M_{\dot\phi}}\delta\left(e^{t/T} + C_1\right) \tag{25}$$

Solving for the general solution for Ω

$$\Omega(t) = \frac{M_\delta}{M_{\dot\phi}}\left(1 + C_1e^{-t/T}\right)\delta \tag{26}$$

Since $\Omega(0)$ =0, C_1 must equal -1 Thus we arrive at the solution found in the lecture notes for $\delta = \delta_a$ for $t > 0$

$$\Omega(t) = \frac{M_\delta}{M_{\dot\phi}}\left(1 - e^{-t/T}\right)\delta_a \tag{27}$$

where $T = I/M_{\dot\phi}$ is the time constant.

Resources

- 18.03, Edwards and Penney, *Elementary Differential Equations with Boundary Value Problems*, 4th ed., Section 1.5. [?]
- 18.03, http://www-math.mit.edu/18.03/, Lecture 3, Discusses first-order linear ODEs and integrating factors. The final example in the lecture notes deals with a problem in the same form as the roll example.

2.4 Laplace Transforms

Required Skills

- Take the Laplace transform and inverse Laplace transform of a function
- Solve linear, constant-coefficient differential equations using Laplace transforms

If you need a review of Laplace transforms please take a look at the following resources.

- 18.03 Course Notes, Sections I, LT. Goes over convolution and the Laplace transform
- 18.03 Lecture Notes, 21-25, Covers the Laplace transform in detail
- 16.06 Van de Vegte, *Feedback Control Systems*, Section 1.6. Gives a quick review of the Laplace transform as well as a table of transforms. [**?**]
- Unified Signals/Systems Spring 2004, Lecture SS8. Provides introduction to the Laplace transform and transfer functions.

3 Lecture 3: Control System Analysis and Design

Lecture 3 Math Topics

- Functions
- Second-Order Ordinary Differential Equations

3.1 Functions

Required Skills

- Understand the concept of an independent and dependent variables (argument of a function)

The functions you will come across for this lecture are $R(s)$, $G_1(s)$, $G_2(s)$, $M(s)$, and $C(s)$. Note that all of these functions are dealing with the Laplace domain.

3.2 Second-order ordinary differential equations

Required Skills

- Convert a linear second-order ODE into a system of first-order ODEs.

For a review of second-order ODEs please refer to the following resource.

- http://www-math.mit.edu/18.03, Class 10. Lecture notes on second-order equations. Includes discussion of spring, mass, damper (dashpot) system.

4 Lecture 4: Disturbances and Sensitivity

4.1 Math Topics

- Differentiation
- Partial Differentiation

4.2 Functions

Required Skills

- Understand the concept of an independent and dependent variable (argument of a function)

The functions you will encounter for this lecture are $G(s)$ and $T(s)$, both in the Laplace domain.

4.3 Differentiation

Required Skills

- Be able to take the derivative of a function.

For a review of basic differentiation definitions please refer to the following resource.

- http://www-math.mit.edu/~djk/18_01/chapter02/contents.html

For a review of basic differentiation formulae, like the quotient rule, please refer to the following resource.

- http://www-math.mit.edu/~djk/18_01/chapter03/contents.html

4.4 Partial Differentiation

Required Skills

- Be able to take the partial derivative of a multivariate function.

For a review of partial differentiation please refer to the following resources.

- 18.02 Course Notes, Section TA, 1. Partial Derivatives
- 18.02 Course Notes, Exercises 2A-2, 2B-9. (The solutions can be found in the same course notes).

5 Lecture 5: Steady-State Errors

5.1 Math Topics

- Limits

5.2 Functions

Required Skills

- Understand the concept of an independent and dependent variable (argument of a function)

The functions you will encounter in lecture 5 include $e(t)$, $E(s)$, $R(s)$, $G(s)$.

5.3 Limits

Required Skills

- Be able to take the limit of some expression with respect to a given variable.

For a review of limits please refer to the following resource.

- http://www-math.mit.edu/~djk/18_01/chapter02/contents.html
- Simmons, George F. *Calculus with Analytic Geometry* 2nd ed.
- If you did not take 18.01 at MIT you may want to refer to your introductory Calculus course textbook.

6 Lecture 6: The s-Plane, Poles and Zeroes

6.1 Math Topics

- Complex Numbers
- Partial Fraction Expansion

6.2 Functions

Required Skills

- Understand the concept of an independent and dependent variable (argument of a function)

The functions you will encounter in lecture 6 are $R(s)$, $G(s)$, $C(s)$, $c(t)$, as well as the exponential function. Like the sine function, the exponential function has a special way of writing it, $e^{j\omega}$ or sometimes $exp(j\omega)$.

6.3 Complex Numbers

Required Skills

- Be able to convert a complex number from Cartesian to polar form and vice versa
- Be able to plot a complex number
- Be able to add and multiply complex numbers
- Be able to simplify a complex fraction using complex conjugation
- Be able to identify the modulus and argument of a complex number

If a complex number z is given as $z = a + bj$ then the modulus and argument of z are defined as

$modulus = \sqrt{a^2 + b^2}$
$\theta = arg(a + bi)$ meaning the argument is the angle the vector from the origin to z makes with the positive real axis.

Finding the modulus is then quite simple because it is just the radius of the complex vector. Finding the argument is also quite simple and is done very easily by visualizing the complex axes. Let's say that $z = 3 + 2j$. Then graphically the complex vector looks like: (next page)

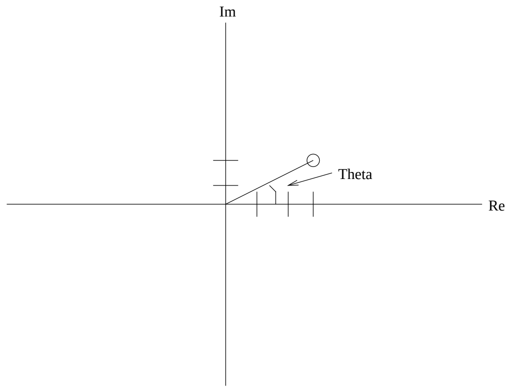

Figure 2: Complex number in the first quadrant

Here it is clear that when the complex vector is in the first quadrant the argument will be simply $\tan^{-1}(2/3)$. If z where instead $z = -3 + 2j$, then the point would be in the second quandrant and the plot would look like:

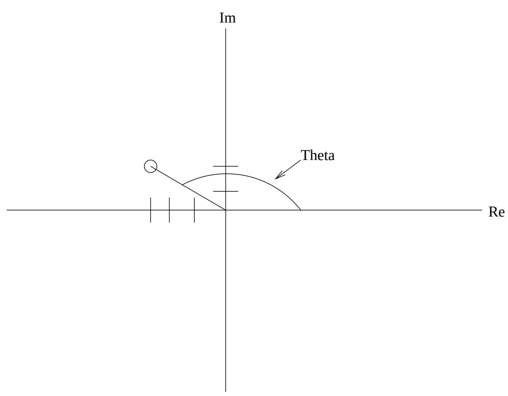

Figure 3: complex number in the second quadrant

Thus the modulus would be the same but the argument would now be $\theta = \tan^{-1}(\frac{2}{-3}) + \pi$ since the argument is the angle taken from the positive real axis.

A note on notation:

Professor Willcox uses the notation $Re^{j\alpha} = R\angle\alpha$ to represent a complex number. So a number written as $5\angle(\pi/4)$ is just a complex number with modulus $= 5$ and argument $= \pi/4$. As in $5e^{j(\pi/4)}$.

For a review of basic complex number theory as well as polar and exponential representations, please refer to the following resources.

- 18.03 Course Notes, Section C. Includes complex arithmetic, polar representation, complex exponentials, and root finding.
- 18.03 Supplementary Notes, Section 7, pg 32. Provides another look at complex numbers.

6.4 Partial Fraction Expansion

Required Skills

- Be able to perform a partial fraction expansion on a fraction

In the lecture notes we are presented with the function,

$$C(s) = \frac{K(s+2)}{s(s+4)} \tag{28}$$

and the inverse Laplace transform of the function is desired. When faced with a function such as $C(s)$ above, it is not immediately apparent what the inverse Laplace transform is. Therefore, we must rewrite the function in a more appropriate form. The method shown in the lecture notes is known as partial fraction expansion, which you were taught in 18.03 as well as 18.01 if you took it at MIT. What follows is a step-by-step approach to the partial fraction expansion of $C(s)$.

Standard Approach to Partial Fractions

$$C(s) = \frac{K(s+2)}{s(s+4)} \tag{29}$$

Rewrite the equation as

$$C(s) = \frac{K_1}{s} + \frac{K_2}{s+4} \tag{30}$$

Thus, we have

$$C(s) = \frac{K(s+2)}{s(s+4)} = \frac{K_1}{s} + \frac{K_2}{s+4}. \tag{31}$$

Now let $K(s+2) = N(s)$ and $s(s+4) = D(s)$ such that $C(s) = N(s)/D(s)$. Multiplying through by $D(s)$, (the denominator of $C(s)$), yields,

$$K(s+2) = \frac{K_1 s(s+4)}{s} + \frac{K_2 s(s+4)}{s+4} = K_1(s+4) + K_2 s \tag{32}$$

Letting $s = 0$ then gives

$$2K = 4K_1 \tag{33}$$

Thus,

$$K_1 = \frac{K}{2} \tag{34}$$

and letting $s = -4$ gives

$$-2K = -4K_2 \tag{35}$$

or

$$K_2 = \frac{K}{2}. \tag{36}$$

Now $C(s)$ can be written as

$$C(s) = \frac{K/2}{s} + \frac{K/2}{s+4}, \tag{37}$$

and the inverse Laplace transform can be easily taken.

Cover-Up Method for Partial Fractions

The cover-up method is a well known technique for partial fraction expansion that makes the process quick and easy. The method is essentially the same as that of the standard approach but it requires about half the work. As an example we will solve the problem we just completed but using the cover-up method this time.

We have,

$$C(s) = \frac{K(s+2)}{s(s+4)} = \frac{K_1}{s} + \frac{K_2}{s+4} \tag{38}$$

The idea now is that we will multiply through by s, leaving

$$\frac{K(s+2)}{(s+4)} = K_1 + \frac{sK_2}{s+4} \tag{39}$$

Now if we let $s = 0$ we are left with

$$\frac{2K}{4} = K_1 = \frac{K}{2} \tag{40}$$

as we found earlier. The beauty of the cover-up method however, lies in the fact that the mathematical manipulations we just did can be done just as easily by covering up certain terms with your hand. As in the example above, to solve for K_1, we know that we must multiply through by s so that we may set $s = 0$. Thus, we know that the K_2 term will drop out, as will the s on the left-hand side of the equation. Thus, we can simply cover-up the $K_2/(s+4)$ term as well as the s term in the denominator of left-hand side, which will leave us with

$$\frac{K(s+2)}{s+4} = K_1 \tag{41}$$

but since we have set $s = 0$, we are really left with $K_1 = K/2$ and we are done.

Let's now solve for K_2 using only the cover-up approach. We know that we are actually going to multiply the equation through by $s+4$ so we may cover up any occurances of $s+4$ in the equation. We also know that when we set $s = -4$ we will eliminate the K_1 term, thus we may cover that up as well. So just by looking at the equation we break the equation down to

$$\frac{K(s+2)}{s} = K_2 \tag{42}$$

and since we have set $s = -4$ we are done and $K_2 = K/2$.

For a further review of partial fraction expansion please refer to the following resources.

- 18.01, http://www-math.mit.edu/~djk/18_01/chapter23/contents.html Includes the general technique as well as the cover up method.
- 18.03, Supplementary Notes, pp. 66-68, section 14.2. Provides an explanation and an example use of the cover up method for partial fraction expansion.
- 18.03, Course textbook, *Elementary Differential Equations with Boundary Value Problems*, 4th ed., Edwards and Penney, Section 4.3. [?]

7 Lecture 7: Transient Response Characteristics and System Stability

7.1 Functions

You will not encounter any new functions in this lecture.

7.2 Complex Numbers and Partial Fractions

For a review of complex numbers and partial fraction expansion, please refer to the resources listed for lecture 6.

8 Lecture 8: Dominant Modes

8.1 Functions

You will not encounter any new functions in this lecture.

8.2 Partial Fractions

Revisit partial fraction expansion again if you need to but you should know it by now.

9 Lecture 9: Transient Performance and the Effect of Zeroes

9.1 Functions

You will not encounter anything new concerning functions.

9.2 Linearization and Second-Order ODEs

Lecture 9 includes material concerning linearization and second-order differential equations. For a review of this material please see the resources from lecture 2 for linearization and those from lecture 3 for second-order ODEs.

10 Lecture 10: The Effect of Zeroes

10.1 Functions

You will not encounter anything new concerning functions.

11 Lecture 11: State Space

11.1 Math Topics

- First-Order Differential Equations
- Second-Order Differential Equations
- Vectors and Matrices

11.2 Functions

Required Skills

- Understand the concept of an independent and dependent variable (argument of a function)

The functions you will encounter in lecture 11 are $\frac{dv}{dt}$, $v(t)$, $\underline{x}(t)$, $\underline{u}(t)$, $\underline{y}(t)$, as well as the derivatives of $\underline{x}(t)$. Note that the underlines on the x, u, and y refer to the fact that these are vector functions.

11.3 First-Order Differential Equations

Required Skills

- Solve a first-order differential equation

For a review of how to solve a first-order differential equation please refer to the math notes for lecture 2 and the resources listed there.

11.4 Second-Order Differential Equations

Required Skills

- Reduce a second-order differential equations to a system of two first order differential equations

For a review of how to reduce a second-order differential equation to a system of first order equations please refer to the following resources.

- 18.03, Course Textbook *Differential Equations with Boundary Value Problems*, 4th ed., Edwards and Penney, section 5.1 examples 4, 5, and 6.
- 18.03, Course notes, section LS2, Linear Systems.

11.5 Vectors and Matrices

Required Skills

- Reduce and equation with several inputs and outputs to vector form
- Write a system of first-order differential equations in vector-matrix form
- Perform a non-singular transformation of a vector

For a review of how to reduce an equation to vector form and how to write a system of first-order equations to vector-matrix form, please refer to the following resources.

- 18.03, Course Textbook *Differential Equations with Boundary Value Problems*, 4th ed., Edwards and Penney, section 5.3.
- 18.03, Course Notes, LS1, Lineaer Systems: Review of Linear Algebra

12 Lecture 12: State Space Modeling

12.1 Math Topics

- N^{th} Order Differential Equations
- Vector-Matrix Equations
- Second-Order Differential Equations

12.2 Functions

Required Skills

- Understand the concept of an independent and dependent variable (argument of a function)

The functions you will encounter in lecture 12 include the n derivatives of $w(t)$, the n derivatives of $x(t)$, $G(s)$, $W(s)$, $R(s)$, $X(s)$, $\underline{y}$, $\underline{x}$.

12.3 N^{th} Order Differential Equations

Required Skills

- Write an n^{th} order differential equation

For a review of how to write out an n^{th} order differential equation please refer to the following resource.

- 18.03, Course Textbook, *Differential Equations with Boundary Value Problems*, 4th ed., Edwards and Penney, p109.

12.4 Vector Matrix Equations

Required Skills

- Write a first-order system of equations in vector-matrix form

For a review of how to write a system of equations in vector-matrix form please refer to the resources listed for that topic in lecture 11.

12.5 Second-Order Differential Equations

Required Skills

- Reduce a second-order differential equation into a system of two first-order differential equations

For a review of how to reduce a second-order equation down to a system of first-order equations please refer to the resources listed for that topic in lecture 11.

13 Lecture 13: More State Space Modeling and Transfer Function Matrices

13.1 Math Topics

- Second-Order Differential Equations
- Vector-Matrix Equations
- Laplace Transforms
- Linear Algebra

13.2 Functions

Required Skills

- Understand the concept of an independent and dependent variable (argument of a function)

The functions you will encounter in lecture 13 include $G(s)$, $R(s)$, $X(s)$, $W(s)$, $\ddot{x}$, $\dot{x}$, $\underline{\ddot{x}}$, $\underline{\dot{x}}$, $\underline{X}(s)$, $Y(s)$, $v(s)$, and $L(s)$.

13.3 Second-Order Differential Equations

Required Skills

- Reduce a second-order differential equation into a system of two first-order differential equations.

Please refer to the list of resources for this topic in lecture 11 for review.

13.4 Vector-Matrix Equations

Required Skills

- Write a system of first-order differential equations in vector-matrix form.

For a review of how to write a system of first-order equations in vector-matrix form please refer to the resources listed in lecture 11.

13.5 Laplace Transforms

Required Skills

- Find the Laplace transform of a differential equation
- Find the Laplace transform of a vector

For a review of how to find the Laplace transform of a differential equation please refer to the following resource.

- 18.03, Course Notes, Section LT.
- 18.03 Lecture Notes 21-25 on OCW.

For a review of how to find the Laplace transform of a vector please look at the lecture 13, 16.06 notes. The notes on page 5 define how this is done.

13.6 Linear Algebra

Required Skills

- Find the inverse of a 2 x 2 matrix
- Add and multiply matrices

For a review of how to find the inverse of a 2 x 2 matrix please refer to the following resource.

- 18.03, Course Notes, Section LS.1, Review of Linear Algebra

For a review of basic matrix math please refer to the following resource.

- 18.03, Course Notes, Section LS.1, Review of Linear Algebra

14 Lecture 14: Quanser Model and State Transition Matrices

14.1 Math Topics

- Second-Order Differential Equations
- First-Order Differential Equations
- Linear Algebra
- Power Series
- Laplace Transforms

14.2 Functions

Required Skills

- Understand the concept of an independent and dependent variable (argument of a function)

The functions you will encounter in lecture 14 include $G(s)$, $\Theta(s)$, $V(s)$, $\dot{\theta}$, $\ddot{\theta}$, $\dot{x}$, $\underline{\dot{x}}$, $\underline{x}$, $\underline{y}$, $\underline{u}$, the exponential function, $\Phi(t)$, $\dot{\Phi}$, and $\Phi(t_2 - t_1)$.

14.3 Second-Order Differential Equations

Required Skills

- Reduce a second-order differential equation to a system of first-order differential equations

For a review of how to do this please refer to the resources listed in lecture 11.

14.4 First-Order Differential Equations

- Solve a first-order differential equation

For a review of how to solve a first-order differential equation please refer to the supplementary notes and resources given for lecture 2.

14.5 Linear Algebra

Required Skills

- Write a system of first-order equations in vector-matrix form
- Find the inverse of a 2 X 2 matrix

For a review of how to write a system of first-order equations in vector-matrix form please refer to the resources given in lecture 11.

For a review of how to find the inverse of a 2 x 2 matrix please refer to the resource listed in lecture 13.

14.6 Power Series

Required Skills

- Write the power series of e^{ax}
- Write the power series of a matrix exponential
- Differentiate a power series

For a review of all of these necessary skills involving power series please refer to the following resource

- 18.03, Course Notes, section LS.6-3, The Exponential Matrix
- 18.03, Course Text Book, *Elementary Differential Equations with Boundary Value Problems*, 4th ed., Edwards and Penney, section 5.7, Exponential Matrices, pp. 416-425.

14.7 Laplace Transforms

Required Skills

- Find the Laplace Transform of a Matrix

For a review of how to find the Laplace Transform of a Matrix please refer to the lecture 14 notes, pages 8 and 9.

15 Lecture 15: Solutions of State Space Differential Equations

15.1 Math Topics

- First-Order Differential Equations
- Integration
- Convolution
- Linear Algebra
- Inverse Laplace Transforms

15.2 First-Order Differential Equations

Required Skills

- Solve a first-order differential equation for homogeneous and particular solutions

For a review of how to solve a first-order equation please refer to the lecture 2 supplementary notes.

15.3 Integration

Required Skills

- Be able to integrate vectors and matrices

For a review of how to integrate please refer to the following resource.

- 18.01, online text, http://www-math.mit.edu/~djk/18_01/contents.html, chapters 21-24.

15.4 Convolution

Required Skills

- Be able to perform a convolution of two functions

For a review of how to convolve functions please refer to the following resource.

- 18.03, Course Notes, Section I.4 Convolution
- 18.03, Course Textbook, *Elementary Differential Equations with Boundary Value Problems*, Edwards and Penney, 4th ed., pp. 292-294.

15.5 Linear Algebra

Required Skills

- Find the inverse of a matrix
- Find the determinant of a matrix

For a review of these linear algebra skills please refer to the resources listed for lecture 13.

15.6 Inverse Laplace Transforms

Required Skills

- Be able to find the inverse Laplace transform of a function

For a review of how to find the inverse Laplace transform of a function please refer to the resources listed for lecture 2.

16 Lecture 16: Controllability

16.1 Math Topics

- First-Order Differential Equations
- Power Series
- Linear Algebra

16.2 First-Order Differential Equations

Required Skills

- Write a first-order system of differential equations in vector/matrix form

For a review of how to do this please refer to the resources listed for lecture 11.

16.3 Power Series

Required Skills

- Write the power series expression for a particular function

For a review of power series please refer to the resources listed for lecture 14.

16.4 Linear Algebra

Required Skills

- Determine whether or not a set of vectors are linearly indepedent

For a review of how to do this please refer to the following resource.

- 18.03, Course Notes, Section LS.4 Linear Independence of Vectors.