

# Fourier Series for Accurate, Stable, Reduced-Order Models for Linear CFD Applications.

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## Abstract

A new method, Fourier model reduction (FMR), for obtaining stable, accurate, low-order models of very large linear systems is presented. The technique draws on traditional control and dynamical system concepts and utilizes them in a way which is computationally very efficient. Discrete-time Fourier coefficients of the large system transfer function are calculated and used to construct the Hankel matrix of an intermediate system with guaranteed stability. Explicit balanced truncation formulae are then applied to obtain the final reduced-order model, whose size is determined by the Hankel singular values of the intermediate system. In this paper, the method is applied to two computational fluid dynamic systems, which model unsteady motion of a two-dimensional subsonic airfoil and unsteady flow in a supersonic diffuser. In both cases, the new method is found to work extremely well. Results are compared to models developed using the proper orthogonal decomposition and Arnoldi method. In comparison with these widely used techniques, the new method is computationally more efficient, guarantees the stability of the reduced-order model, uses both input and output information, and is valid over a wide range of frequencies.

## Introduction

Despite increasing computational resources, for many applications high-order, complicated numerical models are impractical. This is often the

case when coupling between disciplinary models is required. For example, computational fluid dynamic (CFD) models are not appropriate for use in many aeroelastic applications or for active flow control design. The goal of model reduction is to systematically develop a low-order model that captures the relevant system dynamics accurately over a range of frequencies and forcing inputs. For very large systems, the cost of reduction is often a critical issue.

Many effective reduction techniques have been developed in a controls context. An optimal reduced model is one that minimizes the H-Infinity norm of the difference between the reduced and original system transfer functions; however, no polynomial-time algorithm is known to achieve this goal. Algorithms such as optimal Hankel model reduction<sup>1-3</sup> and balanced truncation<sup>4</sup> have been widely used throughout the controls community to generate suboptimal reduced models with strong guarantees of quality. These algorithms can be performed in polynomial time; however, the computational requirements make them impractical for application to large systems such as those encountered in computational fluid dynamic (CFD) applications. In this case, system orders exceed  $10^4$  and the computation of gram-mians is impractical. For this reason, many of the control-based reduction concepts have not been transferred to other disciplines.

The most popular reduction technique for CFD applications is the proper orthogonal decomposition (POD),<sup>5</sup> also known as Karhunen-Loève expansions. Given a set of  $N$  flow solutions, or snapshots,<sup>6</sup> the POD calculates a set of basis vectors that captures the most energetic modes in the flow. This set is optimal in the sense that the least squares error between the original snapshot ensemble and its reconstruction in the space spanned by the first  $k$  POD vectors is minimized for all  $k \leq N$ . The widely-used approach to model

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reduction is to project the CFD governing equations onto the space spanned by the first  $k$  POD vectors. While the reconstruction of the original snapshots might be "optimal" using the POD basis, one can make no statement about the accuracy of the resulting reduced-order dynamical system. In fact, one cannot even guarantee the stability properties of this system. Despite this, the POD is found to work well in many cases has been successfully applied for a broad range of CFD applications (see, for example, reviews in Dowell and Hall<sup>7</sup> and Beran and Silva<sup>8</sup>).

Another class of reduction techniques that are based on matching moments of the system transfer function has been developed for analysis of large linear circuits. For instance, the Arnoldi algorithm can be used to generate vectors which form an orthonormal basis for the Krylov subspace, and has been applied to analysis of RLC circuits,<sup>9</sup> turbomachinery aeroelastic behavior<sup>10</sup> and active control design for a supersonic diffuser.<sup>11</sup> The Arnoldi method is computationally more efficient than the POD, but also offers no guarantees as to the accuracy or stability of the reduced-order model. Moreover, neither the POD nor Arnoldi takes account of system outputs when performing the reduction, hence the reduced-order models produced may be inefficient. The POD has been suggested as a means to obtain an approximate balanced truncation for large systems using both inputs and outputs;<sup>12,13</sup> however, the reduction approach is computationally expensive and again offers no stability guarantees.

In Willcox and Megretski,<sup>14</sup> a new technique, Fourier model reduction (FMR), for performing model reduction of very large systems was presented. This method draws on classical dynamical system and control theory concepts, and applies them using an iterative procedure that is very efficient for large systems. The resulting reduced-order models are guaranteed to preserve the stability properties of the original system and have an associated error bound that depends on the smoothness of the original transfer function. In this paper, the FMR algorithm is reviewed and then applied to two different CFD applications, both of which require fluid models of low order. The first example investigates the unsteady motion of a two-dimensional subsonic airfoil, which

forms the fluid component of an aeroelastic analysis. The second example derives a model of the flow dynamics of a supersonic inlet that will be used to derive active control strategies and is used to demonstrate the multiple-input, multiple-output (MIMO) capabilities of the FMR algorithm. A comparison is also made with reduced models derived using Arnoldi and POD.

The outline of the paper is as follows. In the following section, the dynamical system arising from implementation of CFD is briefly described. The FMR technique is then presented, along with the extension to the MIMO case, and compared with Arnoldi and POD approaches. The two CFD examples are then considered, and finally, conclusions are drawn.

### Computational Fluid Dynamic Model

Consider a general linearized CFD model, which can be written as

$$G : \quad E \frac{d}{dt} x = Ax + Bu, \\ y = Cx + Du \quad (1)$$

where  $x(t) \in \mathbf{R}^n$  contains the  $n$  unknown perturbation flow quantities at each point in the computational grid. For example, for two-dimensional, compressible, inviscid flow, which is governed by the Euler equations, the unknowns at each grid point are the perturbations in flow density, Cartesian momentum components and flow energy. The vectors  $u(t)$  and  $y(t)$  in (1) contain the system inputs and outputs respectively. The definition of inputs and outputs will depend upon the problem at hand. In aeroelastic analysis of a wing, inputs consist of wing motion while outputs of interest are the forces and moments generated. For control purposes, the output might monitor a flow condition at a particular location which varies in response to a disturbance in the incoming flow.

The linearization matrices  $A$ ,  $B$ ,  $C$ ,  $D$  and  $E$  in (1) are evaluated at steady-state conditions. The matrix  $E$  is included for generality, and may contain some zero rows, which arise from implementation of flow boundary conditions. On solid walls, a condition is imposed on the flow velocity, while at farfield boundaries certain flow parameters are specified, depending on the nature of the boundary (inflow/outflow) and the local flow conditions (subsonic/supersonic). Although these

prescribed quantities could be condensed out of (1) to obtain a smaller state-space system, such a manipulation is often complicated and can destroy the sparsity of the system. The more general form of the system is therefore considered.

The system (1) is efficient for time computations since a time discretization, such as backward Euler, can be applied and the resulting large  $n \times n$  system matrix inverted just once. However, the order of the system is still prohibitively high for many applications, such as aeroelasticity and active flow control. In the next section, we present an efficient method with quality guarantees to reduce the size of the system while retaining an accurate representation of important flow dynamics.

### Fourier Model Reduction

We consider the task of finding a low-order, stable, continuous time, linear time invariant (LTI), state-space model

$$\hat{G} : \begin{aligned} \frac{d}{dt} \hat{x}(t) &= \hat{A} \hat{x}(t) + \hat{B} u(t), \\ \hat{y}(t) &= \hat{C} \hat{x}(t) + \hat{D} u(t) \end{aligned} \quad (2)$$

which approximates well the given stable state-space model (1). We consider first the case of single input, single output (SISO), then discuss later the MIMO case. Typically,  $A$  and  $E$  in (1) are sparse, square matrices of very large dimension  $n > 10^4$ , and the desired order  $k$  of  $\hat{G}$  is less than 50.

The quality of  $\hat{G}$  as an approximation of  $G$  is defined as the H-Infinity norm of the difference between their transfer functions:

$$\|\hat{G} - G\|_{\infty} = \sup_{\omega \in \mathbf{R}} |\hat{G}(j\omega) - G(j\omega)|, \quad (3)$$

which in turn equals the square root of the maximal energy

$$\|\hat{y} - y\|_2^2 = \int_t |\hat{y}(t) - y(t)|^2 dt \quad (4)$$

of the difference  $e = \hat{y} - y$ , which can be generated when testing both  $G$  and  $\hat{G}$  with an arbitrary unit energy input  $u$  as shown in Figure 1. With this measure of model reduction error, if a good reduced model  $\hat{G}$  is found,  $G$  can be represented, for design or analysis purposes, as a series connection (i.e. a sum) of  $\hat{G}$  and a small "uncertain"

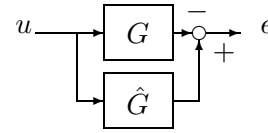


Fig. 1 Comparing  $G$  and  $\hat{G}$

error system  $\Delta = G - \hat{G}$ , and the standard results from robustness analysis can be applied to predict the effect of replacing  $G$  with  $\hat{G}$  in even larger scale systems.

FMR is a low complexity algorithm that allows one to perform model reduction of very large systems. While the result is not an optimal reduced model of  $G$ , it satisfies an attractive guaranteed H-Infinity bound.

### Fourier Series of Discrete Time Systems

Consider the full discrete time (DT) LTI system model  $g$  defined by the difference equations

$$g : \begin{aligned} x(t+1) &= ax(t) + bu(t) \\ y(t) &= cx(t) + du(t), \end{aligned} \quad (5)$$

where  $a, b, c, d$  are given matrices of coefficients,  $x(t) \in \mathbf{R}^n$  is the system state, and  $u(t), y(t)$  are scalar input and output. It will be assumed that  $g$  is stable, i.e.  $\rho(a) < 1$ , where  $\rho(M)$  denotes the spectral radius of  $M$ , defined as the maximal absolute value of its eigenvalues.

The transfer function

$$g(z) = d + c(zI - a)^{-1}b \quad (6)$$

has the Fourier decomposition

$$g(z) = \sum_{k=0}^{\infty} g_k z^{-k}, \quad (7)$$

where

$$g_0 = d, \quad g_k = ca^{k-1}b \quad (k = 1, 2, \dots) \quad (8)$$

The Fourier expansion converges exponentially for  $|z| > \rho(a)$ . Note that the first  $m$  Fourier coefficients  $g_k$  are easy to calculate using the "cheap" iterative process

$$g_k = ch_{k-1}, \quad h_k = ah_{k-1} \quad (k = 1, \dots, m), \quad (9)$$

where  $h_0 = b$ ,

which is expected to be "stable" since  $\rho(a) < 1$ .

Let  $\hat{g}_m$  denote the  $m^{\text{th}}$  order approximation of  $g$  based on the Fourier series expansion:

$$\hat{g}_m(z) = \sum_{k=0}^m g_k z^{-k}. \quad (10)$$

We note that the approximation  $\hat{g}_m$  is guaranteed to be stable and that the error is related to the smoothness of  $g$  as

$$\|g - \hat{g}_m\|_\infty^2 \leq \frac{m^{1-2q}}{2\pi(2q-1)} \int_{-\pi}^{\pi} |g^{(q)}(e^{j\tau})|^2 d\tau \quad (11)$$

for  $q = 1, 2, \dots$ , where  $g^{(q)}$  is the  $q^{\text{th}}$  derivative of  $g$  with respect to  $\tau$ .<sup>14</sup>

### Fourier Series of Continuous Time Systems

Consider the full continuous time LTI system model  $G$  defined by the system (1), where  $u(t), y(t)$  are scalar input and output. If  $E$  is invertible then  $x = x(t)$  is the system state. It will be assumed that  $G$  is stable, i.e. that all roots of the characteristic equation  $\det(sE - A) = 0$  have negative real part, and that  $C(sE - A)^{-1}B$  remains bounded as  $s \rightarrow \infty$ .

Let  $\omega_0 > 0$  be a fixed positive real number. The transfer function

$$G(s) = D + C(sE - A)^{-1}B \quad (12)$$

has the Fourier decomposition

$$G(s) = \sum_{k=0}^{\infty} G_k \left( \frac{s - \omega_0}{s + \omega_0} \right)^k, \quad (13)$$

where

$$G_0 = d, \quad G_k = ca^{k-1}b \quad (k = 1, 2, \dots), \quad (14)$$

$$d = D + C(\omega_0 E - A)^{-1}B, \quad (15)$$

$$a = -(\omega_0 E + A)(\omega_0 E - A)^{-1}, \quad (16)$$

$$c = 2\omega_0 C(\omega_0 E - A)^{-1}, \quad (17)$$

$$b = -E(\omega_0 E - A)^{-1}B. \quad (18)$$

The Fourier expansion is derived from the identity

$$G(s) = g(z) = d + c(zI - a)^{-1}b \quad \text{for } z = \frac{s + \omega_0}{s - \omega_0},$$

which allows one to apply the observations and theorem from the previous subsection to this case. Note that by comparing (7) and (13), it can be seen that  $G_k = g_k$ .

### Reduced Model Construction

To construct an  $m^{\text{th}}$ -order reduced model, one first calculates the Fourier coefficients,  $g_0, g_1, \dots, g_m$ . The DT reduced model is then given by

$$\hat{g} : \quad \begin{aligned} \hat{x}[t+1] &= \hat{a}\hat{x}[t] + \hat{b}u[t] \\ \hat{y}[t] &= \hat{c}\hat{x}[t] + \hat{d}u[t], \end{aligned} \quad (19)$$

where

$$\hat{a} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & \ddots & \dots \end{bmatrix} \quad \hat{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad (20)$$

$$\hat{c} = [g_1 \quad g_2 \quad \dots \quad g_m] \quad \hat{d} = g_0.$$

An effective approach is to use the efficient iterative procedure to calculate several hundred coefficients, resulting in an intermediate reduced model of the form (19). A second reduction step using balanced truncation can now be performed easily, since the expressions for the grammians are known explicitly. For the DT reduced model (19), the controllability matrix is the identity matrix and the observability matrix is the Hankel matrix that has  $\hat{c}$  as its first row. The balancing vectors can therefore be obtained by computing the singular vectors of the  $m^{\text{th}}$ -order Hankel matrix

$$\Gamma = \begin{bmatrix} g_1 & g_2 & g_3 & \dots & g_{m-1} & g_m \\ g_2 & g_3 & g_4 & \dots & g_m & 0 \\ g_3 & g_4 & g_5 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ g_{m-1} & g_m & 0 & \dots & 0 & 0 \\ g_m & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \quad (21)$$

The Hankel singular values,  $\sigma_i$ ,  $i = 1, 2, \dots, m$ , of the intermediate reduced system are given by the singular values of  $\Gamma$ .

The final reduced system,  $\tilde{g}$  is computed as

$$\tilde{a} = V_k' \hat{a} V_k, \quad \tilde{b} = V_k' \hat{b}, \quad \tilde{c} = \hat{c} V_k, \quad \tilde{d} = \hat{d} \quad (22)$$

where  $V_k$  is a matrix whose columns are the first  $k$  singular vectors of  $\Gamma$ , and  $k$  is chosen according to the distribution of Hankel singular values. Specifically, it can be shown that the following error bound holds:

$$\|\hat{G} - \tilde{G}\|_\infty \leq 2 \sum_{i=k+1}^m \sigma_i. \quad (23)$$

It should be noted that the application of optimal Hankel model reduction to the intermediate reduced model would yield a bound on the error lower than that in (23); however, balanced truncation can be applied very efficiently since the grammians are known explicitly. This also avoids the explicit construction of the intermediate system, as can be seen in the following algorithm.

#### Fourier Model Reduction Algorithm

The Fourier model reduction algorithm is summarized in the following steps.

1. Choose a value of  $\omega_0$ . The value of  $\omega_0$  should reflect the frequency range of interest. The nominal value is unity; however, if the response at high frequencies is of interest, a higher value of  $\omega_0$  should be chosen. One can visualize the transformation from continuous to discrete frequency as a mapping of the imaginary axis in the  $s$ -plane to the unit circle in the  $z$ -plane. The value of  $\omega_0$  then describes the compression of frequencies around the unit circle.
2. Calculate  $m + 1$  Fourier coefficients using (14)-(18). Using the iterative procedure, any number of coefficients can be calculated with a single  $n^{\text{th}}$ -order matrix inversion or factorization.
3. Using (21), calculate the  $m^{\text{th}}$ -order Hankel matrix. Calculate its singular values and singular vectors.
4. Using (22), construct a  $k^{\text{th}}$ -order DT system,  $\tilde{g}$ . The value of  $k$  is chosen according to the distribution of Hankel singular values of the intermediate system.
5. Convert the  $k^{\text{th}}$ -order, DT reduced model to a continuous-time model using the relationships

$$\hat{A} = \omega_0 (\hat{a} - I)^{-1} (\hat{a} + I), \quad (24)$$

$$\hat{B} = 2\omega_0 (\hat{a} - I)^{-1} \hat{b}, \quad (25)$$

$$\hat{C} = -\hat{c} (\hat{a} - I)^{-1}, \quad (26)$$

$$\hat{D} = \hat{d} - \hat{c} (\hat{a} - I)^{-1} \hat{b} \quad (27)$$

#### Multiple-Input, Multiple-Output Systems

Although the above algorithm was derived for SISO systems, it can be easily extended to the

MIMO case. In fact, the steps above are essentially unchanged. Consider the general case of  $p$  inputs and  $q$  outputs. For Step 2, one can still use (14)-(18) to calculate the Fourier coefficients, although now each  $g_k$  is a  $q \times p$  matrix. The iterative procedure is still very efficient, since consideration of multiple inputs and outputs requires only additional matrix vector products.

In Step 3, one again uses (21) to construct the Hankel matrix. The size of this matrix will now be  $m q \times m p$ . The balanced truncation in Step 4 proceeds as before, noting only that each scalar entry of  $\hat{a}$  and  $\hat{b}$  in (20) is replaced with a block entry (zero matrix or identity matrix, as appropriate) of size  $p \times p$ . Finally, the transformations in Step 5 are valid for the MIMO case.

In the results section, two examples will be presented that demonstrate the effectiveness of this approach. We first compare the advantages of FMR with other commonly used techniques.

#### Comparison with Alternative Reduction Methods

The POD is the most widely used reduction method in the fluid dynamics community.<sup>5</sup> The basic idea is to collect a set of snapshots, which are solutions of the linearized CFD model at selected time instants or frequencies. These snapshots are then combined to form an efficient basis, which is optimal in the sense that it minimizes the error between the exact and projected CFD data. While the method results in reduced-order models which accurately reproduce those dynamics included in the sampling process, the models are not valid outside the sampled range. If snapshots are obtained in the time domain, it can be difficult to choose an appropriate forcing function and typically many snapshots are required to obtain accurate results. Using the frequency domain to compute snapshots is often more convenient, however each frequency sampled requires an  $n^{\text{th}}$ -order matrix inversion. Moreover, the reduced-order models obtained via the POD offer no guarantee of stability.

The Arnoldi method has also been used for model reduction of large CFD systems.<sup>10</sup> The goal of moment-matching techniques, such as the Arnoldi method, is to determine a reduced-order model by matching moments (or Taylor series coefficients) of the high-order system transfer function. The Arnoldi method has an advantage over

the POD in that a sequence of basis vectors can be generated in the frequency domain with just a single  $n^{\text{th}}$ -order matrix inversion, and the approach of generating an intermediate model which can be subsequently further reduced via Hankel model reduction also works effectively for this technique. However, the Arnoldi method does not consider system outputs when forming the basis vectors and the resulting reduced-order models are not guaranteed to be stable.

Since the Arnoldi basis vectors are derived from an expansion about zero frequency, for some applications the resulting reduced-order model can be large if frequencies far away from zero are of interest. The multiple frequency point Arnoldi method attempts to address this issue by considering transfer function expansions about multiple frequency points.<sup>11</sup> In this way, accurate models can be derived over a specified frequency range. The reduction cost in this case is proportional to the number of frequency points considered. By generating multiple Arnoldi vectors at each frequency point, the samples can be placed further apart without loss in accuracy. The multi-point Arnoldi method therefore provides a way to trade between the low reduction cost of Arnoldi and the frequency span of POD.

Table 1 compares the attributes of each of these reduction techniques with the new FMR approach described in this paper. Reduction cost refers to the number of  $n^{\text{th}}$ -order system inversions required. (This is the dominating factor in reduction of large CFD systems.) Frequency range refers to the range of validity of the resulting reduced-order models. This range is selected a priori for the POD and multi-point Arnoldi approaches. The validity of the Arnoldi-based reduced-order model is restricted to frequencies close to the expansion point (usually taken to be zero), while the new approach allows a wider range of frequencies to be considered through the choice of  $\omega_0$ . Finally, we note that none of the alternative methods use information pertaining to system outputs when deriving the reduced model; moreover, none are guaranteed to produce a stable reduced-order model. In practice, the POD and Arnoldi approaches can often generate unstable models even though the original system is stable.

The new methodology outperforms other tech-

niques in all categories listed in Table 1. The accuracy of the FMR reduced model can also be quantified if the smoothness of the original system transfer function is known. We now present two test cases that demonstrate the new methodology. Results will also be shown to compare the new method against Arnoldi and POD.

## Results

Two CFD applications will be considered. Each has very different flow dynamics and uses a different CFD formulation; however, FMR will be shown to work very effectively for both.

### Subsonic Airfoil

The first example is a two-dimensional NACA 0012 airfoil operating in unsteady plunging motion with a steady-state Mach number of 0.755. The flow is assumed to be inviscid, so the governing equations are the linearized Euler equations, which have four unknowns per grid point. A finite volume CFD formulation is used<sup>15</sup> with a CFD mesh containing 3482 grid points, which corresponds to a total of  $n = 13928$  unknowns in the linear state-space system. The input to this system is a rigid plunging motion (vertical motion of the airfoil), while the output of interest is the lift force generated. This input and output are typical for an aeroelastic analysis, which typically would also include pitching (angular) motion as an input and airfoil pitching moment as an output.

Fourier coefficients were generated using (14)-(18) for several different values of  $\omega_0$ . The first 201 Fourier coefficients are plotted in Figure 2 for  $\omega_0 = 1$ . For demonstration purposes, the  $m = 200$  intermediate models for  $\omega = 1, 5, 10$  were constructed and the resulting transfer functions are compared with the CFD in Figure 3. One would not expect the plunge dynamics to require such a large number of states; however, since the cost of computing additional Fourier coefficients is small and the process is guaranteed to be stable, one can choose  $m$  to be large. As (11) shows, as the number of coefficients increases, the reduction error decreases. Indeed, as can be seen in Figure 3, the transfer functions match very closely over a large frequency range. Since the original CFD model is stable, these reduced-order models are also all guaranteed to be stable. Moreover, the choice of  $\omega_0$  does not have a large effect in this

Table 1 Comparison of reduction techniques for CFD systems.

Method	Reduction Cost	Frequency Range	Stability Preserved	Reduction Considers
Time POD	$O(1)$	Selected	No	Inputs
Frequency POD	$O(\text{No. of snapshots})$	Selected	No	Inputs
Arnoldi	$O(1)$	Restricted	No	Inputs
Multi-Point Arnoldi	$O(\text{No. of freq. points})$	Selected	No	Inputs
FMR	$O(1)$	More general	Yes	Inputs and Outputs

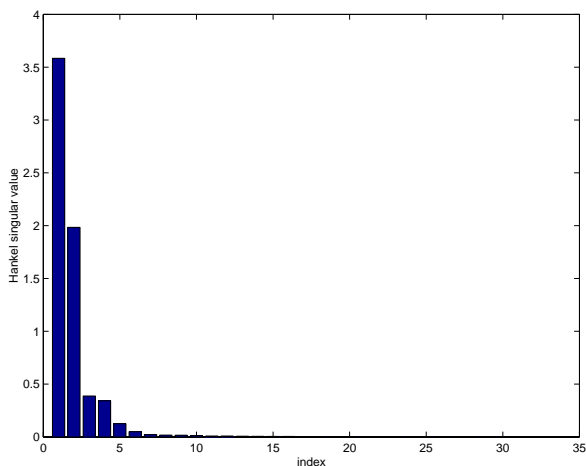


Fig. 2 First 201 Fourier coefficients of the transfer function from plunging motion to lift force for subsonic airfoil.  $\omega_0 = 1$ .

case.

Figure 3 is shown to demonstrate the methodology; in practice these intermediate models would not be computed. Rather, the Fourier coefficients are used to construct the Hankel matrix for further reduction via balanced truncation. The first thirty Hankel singular values of the intermediate  $\omega_0 = 1$  system are shown in Figure 4. These data indicate that a further reduction to five states can be achieved with virtually no loss in accuracy. The transfer function of a five-state model is shown in Figure 5, and can be seen to match the CFD data extremely well.

In order to compare the performance of the new method, reduced-order models for this problem were computed using the POD and Arnoldi methods. POD snapshots were obtained by causing the airfoil to plunge in sinusoidal motion at selected frequencies. Frequencies were selected at 0.1 increments from  $\omega = 0$  to  $\omega = 2.0$ , requiring 21 complex inversions and solves. From these snapshots a set of POD basis vectors was obtained.

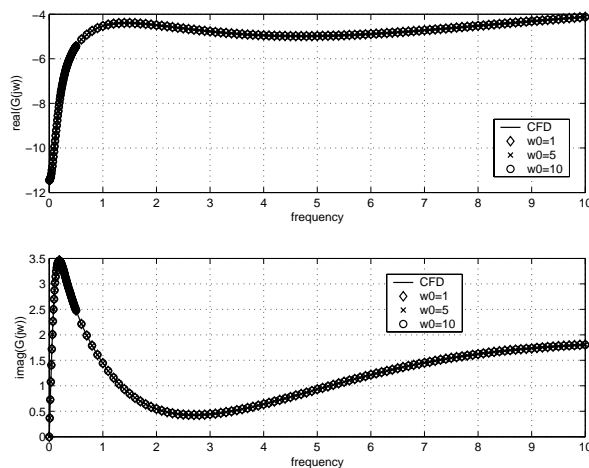


Fig. 3 Transfer function from plunging motion to lift force for subsonic airfoil. Results from CFD model ( $n = 13,928$ ) are compared to reduced-order models derived using 201 Fourier coefficients with  $\omega_0 = 1, 5, 10$ .

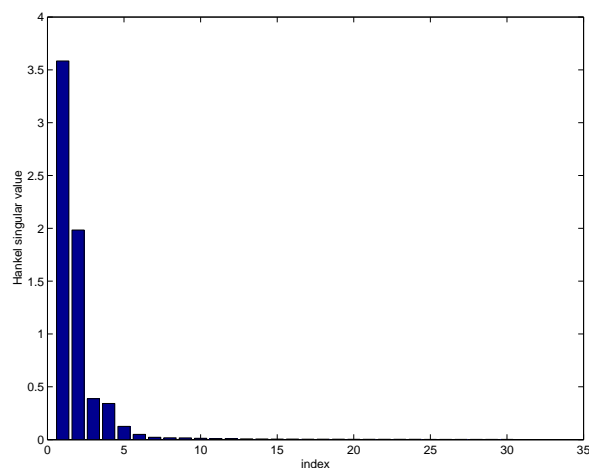
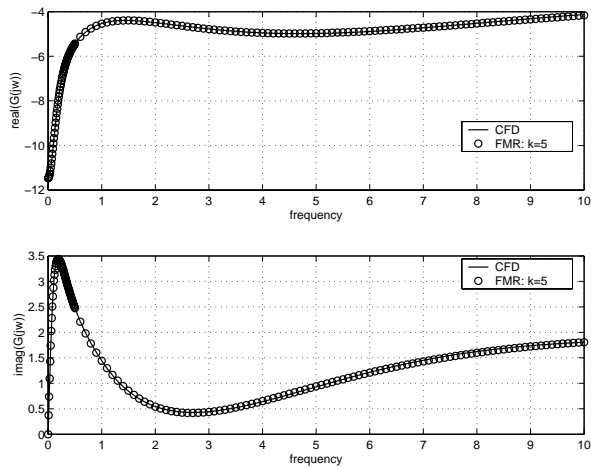


Fig. 4 Hankel singular values for subsonic airfoil case. Values are calculated using 201 Fourier coefficients, evaluated with  $\omega_0 = 1$ .



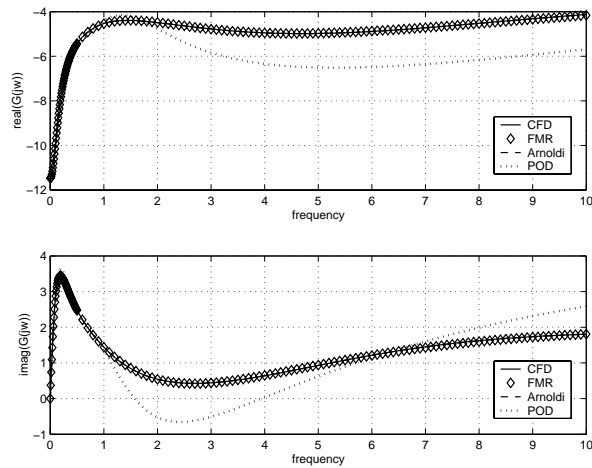
**Fig. 5** Transfer function from plunging motion to lift force for subsonic airfoil. Results from CFD model ( $n = 13,928$ ) are compared to a  $k = 5$  FMR model derived using  $m = 200$ ,  $\omega_0 = 1$ .

Arnoldi vectors were generated about  $\omega = 0$ . Using the same approach as for the new method, 200 Arnoldi vectors were generated (with the cost of a single  $n^{\text{th}}$ -order inversion), and balanced truncation was applied to the resulting  $200^{\text{th}}$ -order system. Figure 6 shows the transfer functions of fifth-order reduced-order models constructed using POD, Arnoldi and the new FMR approach. It can be seen that even though it is much more expensive to compute than the other methods, the POD has the worst performance. In particular, the figure shows that, as expected, the POD-based reduced-order model has a large error outside the frequency range included in the snapshots. For this example, the Arnoldi-based reduced-order model shows similar accuracy to the new approach.

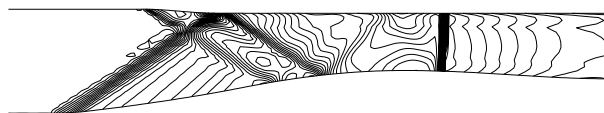
### Supersonic Diffuser

For the second example, we consider unsteady flow through a supersonic diffuser as shown in Figure 7. The CFD model is developed by linearizing the Euler equations about a steady-state solution, the Mach contours for which are plotted in Figure 7. The nominal inflow Mach number for this steady case is 2.2. The CFD formulation for this problem results in a system of the form (1) and is described fully in Lassaux.<sup>16</sup> The CFD model has 3078 grid points and 11,730 unknowns.

A reduced-order model is required for the diffuser dynamics in order to derive active control strategies. These strategies will be used to coun-



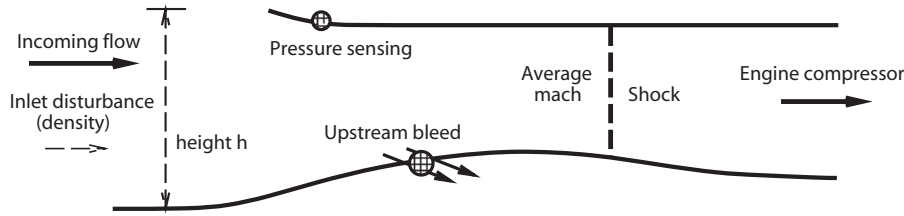
**Fig. 6** Transfer function from plunging motion to lift force for subsonic airfoil. Results from CFD model ( $n = 13,928$ ) are compared to  $k = 5$  reduced-order models derived using FMR, Arnoldi and POD. Both the Arnoldi and FMR models have been created using balanced truncation from an  $m = 200$  system.



**Fig. 7** Mach contours for steady flow through supersonic diffuser. Steady-state inflow Mach number is 2.2.

teract the effect of variations in the incoming flow. In nominal operation, there is a strong shock downstream of the diffuser throat, as can be seen in Figure 7. Incoming disturbances can cause the shock to move forward towards the throat. When the shock sits at the throat, the inlet is unstable, since any disturbance that moves the shock slightly upstream will cause it to move forward rapidly, leading to unstart of the inlet. This is extremely undesirable, since unstart results in a large loss of thrust. In order to prevent unstart from occurring, flow bleeding upstream of the diffuser throat will be used to actively control the position of the shock as shown in Figure 8.

There are therefore two inputs of interest to the system: the incoming flow perturbation and the bleed actuation. For this example, the flow perturbation of interest is a variation in density that is spatially constant across the diffuser in-

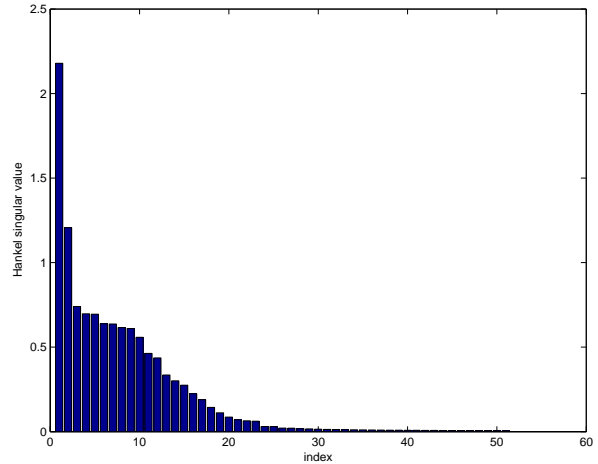


**Fig. 8 Supersonic diffuser active flow control problem setup. Upstream bleed is used to control the position of the shock in the presence of incoming flow disturbances.**

let plane. The output of interest is the average Mach number at the throat, which can be monitored to ensure that the shock remains sufficiently far downstream. The reduced-order model should capture the dynamics accurately for frequencies  $f < 2f_0$ , where the reference frequency  $f_0$  is defined as the freestream speed of sound divided by the diffuser height. This frequency range is sufficient to capture typical disturbances due to atmospheric variations.

In Willcox and Megretski,<sup>14</sup> the FMR technique was applied to each of these transfer functions independently. It was found that ten states were sufficient to capture the bleed dynamics accurately, but more states were required for the disturbance dynamics since they contain a delay. In the latter case, at least twenty states were required to obtain a good fit for  $f < 2f_0$  and values of  $\omega_0 = 5, 10$  yielded more accurate models than  $\omega_0 = 1$ . The FMR models for each set of dynamics represent a considerable reduction in order from the original CFD system, however it is desirable to obtain a single reduced-order model that captures the effects of both inputs.

The FMR algorithm was applied to this two-input, single-output problem with  $\omega_0 = 5$ .  $m = 201$  sets of Fourier coefficients were calculated, where each set contained two coefficients (one corresponding to each input/output pair). The  $200 \times 400$  Hankel matrix was then constructed and used to perform a further reduction via balanced truncation. The first fifty Hankel singular values of the intermediate system are plotted in Figure 9. Based on this distribution, twenty states were retained for the final two-input, single-output reduced-order model. The transfer functions of this twenty-state model are compared with CFD results in Figures 10 and 11. As the figures show, with only twenty states the MIMO reduced-order model captures the dynamics very accurately over the frequency range of interest.

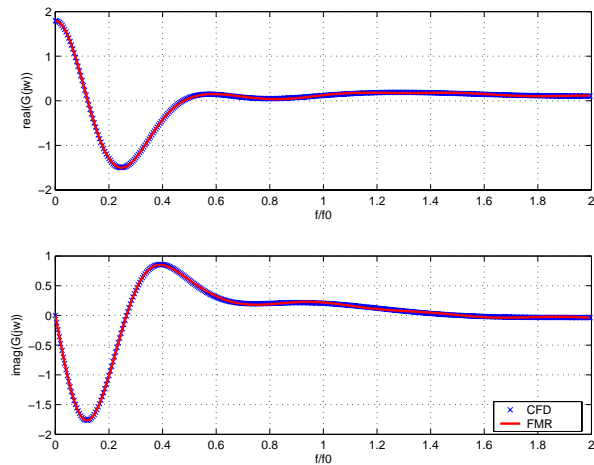


**Fig. 9 Hankel singular values of intermediate FMR supersonic diffuser system. 201 coefficients were used for each input and  $\omega_0 = 5$ .**

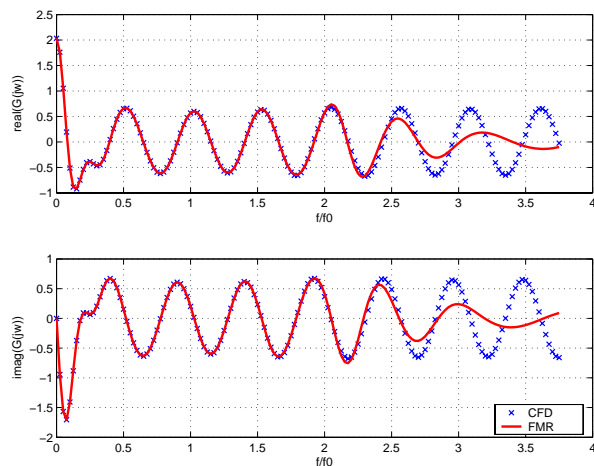
If a better match at higher frequencies is required for the density disturbance dynamics, more Fourier coefficients could be computed to form the intermediate model, or a higher value of  $\omega_0$  could be chosen.

Finally, a comparison is made for the bleed dynamics using FMR, Arnoldi and POD models, each of size  $k = 10$ . A ten-state SISO FMR model was derived using  $m = 201$  coefficients and  $w_0 = 5$ . For the Arnoldi method, the two step reduction approach could not be used in this case, since including a high number of Arnoldi vectors in the basis resulted in an unstable intermediate model. The Arnoldi reduced model was therefore computed by projecting the CFD model onto the space spanned by the first ten Arnoldi vectors. The POD model was obtained by computing 41 snapshots at 21 equally-spaced frequencies from  $f/f_0 = 0$  to  $f/f_0 = 2$ . This required the inversion of one real and 20 complex  $n^{th}$ -order matrices. The resulting transfer functions are plotted in Figure 12.

It can be seen from Figure 12 that the FMR

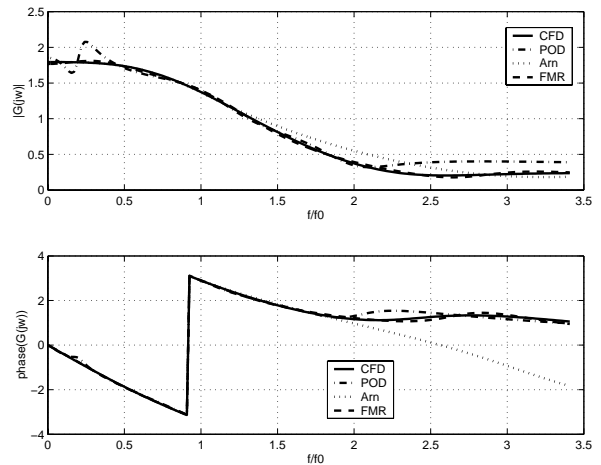


**Fig. 10** Transfer function from bleed actuation to average throat Mach number for supersonic diffuser. Results from CFD model ( $n = 11,730$ ) are compared to MIMO FMR reduced-order model with twenty states.



**Fig. 11** Transfer function from incoming density perturbation to average throat Mach number for supersonic diffuser. Results from CFD model ( $n = 11,730$ ) are compared to MIMO FMR reduced-order model with twenty states.

model matches the CFD results well over the entire frequency range plotted, with a small discrepancy at higher frequencies. The Arnoldi model matches well for low frequencies, but shows considerable error for  $f/f_0 > 1.3$ . The POD model has some undesirable oscillations at low frequencies, and strictly is only valid over the frequency range sampled in the snapshot ensemble ( $f/f_0 < 2$ ). The performance of the POD and Arnoldi models can be improved by increasing the size of the reduced-order models, however, in prac-



**Fig. 12** Transfer function from bleed actuation to average throat Mach number for supersonic diffuser. Results from CFD model ( $n = 11,730$ ) are compared to FMR, POD and Arnoldi models with  $k = 10$  states.

tice, it is found that if too many basis vectors are included, these techniques result in unstable models. For the POD, one could also consider adding more snapshots to the ensemble. A major disadvantage of these methods is that the choice of the number of POD or Arnoldi basis vectors to be included and the snapshot locations must be determined in an ad hoc manner that balances stability and accuracy. Moreover, as in the diffuser flow case considered here, stability of the reduced model is often a troublesome issue.

## Conclusions

Fourier model reduction (FMR) is a new method for model reduction of very large linear systems. The method yields accurate, guaranteed stable reduced-order models, which can be derived using an efficient iterative procedure. An effective use of the method is to derive the Hankel matrix of an intermediate reduced system, with more states than desired but which captures the relevant dynamics very accurately. Balanced truncation can then be applied efficiently to obtain a final reduced-order model whose size is chosen according to the distribution of Hankel singular values.

Results have been presented for two large dynamical systems, arising from implementation of computational fluid dynamic methods. While the flow dynamics of the two systems are very different - one is a subsonic external flow, the other

a supersonic internal flow - the new method is shown to work extremely effectively in both cases. The cost to evaluate the reduced-order models is much lower than for other commonly used methods such as the proper orthogonal decomposition. Moreover, the new method yields reduced-order models that are not restricted to a particular frequency range, account for system inputs and outputs, and have guaranteed stability.

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