Domain Decomposition Preconditioners for Higher-order Hybridizable Discontinuous Galerkin Discretizations

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Outline

1. Introduction and Motivation
2. Background
3. Scalar Advection-Diffusion Equation
4. Linearized Navier-Stokes Equations
5. Summary and Conclusions
Motivation: Developing Scalable CFD

Objective: Perform high-fidelity CFD simulations using high-performance computing in similar amount of time as typical industrial simulations on commodity hardware

- High performance computing in aerospace
  - Typical large jobs remain in O(100) processors
  - “The scalability of most of [CFD] codes tops out around 512 cpus...” – Mavriplis et al [2007]
  - Algorithmic improvements are needed to scale to > 10,000 cpus

- Domain decomposition for elliptic problem
  - Independent local solvers on each processor
  - Coupled global solver
  - Small number of globally coupled DOF
  - Algorithms have been developed and used on > 100,000 cpus

Can we use domain decomposition theory for elliptic problems to develop better solvers for Euler/Navier-Stokes problems?
Approach: HDG/BDDC

- Hybridizable discontinuous Galerkin (HDG) discretization
  - Solution represented by piecewise polynomials on each element
  - Implicit
  - Higher-order solutions
  - Unstructured meshes
  - Well suited to hp-adaptation

- Balancing Domain Decomposition by Constraints (BDDC)
  - Preconditioner for the Schur complement problem
  - Developed by Dohrmann [2003] for structural mechanics applications
  - Makes use of the finite element assembly of the system matrix
  - Condition number bound $\kappa < C(1 + \log(H/h))^2$ for second order elliptic problems
  - Coarse space is defined algebraically using discrete harmonic functions
HDG Discretizations

- Governing equation:
  \[ \nabla \cdot (F(u) + G(u, \nabla u)) = f \]
  - \( u \): state, \( F(u) \): inviscid flux, \( G(u) \): viscous flux

- Local Solver:
  \[ (q_h, v)_\kappa + (u_h, \nabla \cdot v)_\kappa - \langle \hat{u}_h, v \cdot n \rangle_{\partial \kappa} = 0, \quad \forall v \in P^p(\kappa) \]
  \[ -(F + G, \nabla w)_\kappa + \left\langle (\hat{F}_h + \hat{G}_h) \cdot n, w \right\rangle_{\partial \kappa} = (f, w)_\kappa, \quad \forall w \in P^p(\kappa) \]

- Numerical Fluxes:
  \[ (\hat{F}_h + \hat{G}_h) \cdot n = (F(\hat{u}_h) + G(\hat{u}_h, q_h)) \cdot n + S(\hat{u}_h)(u_h - \hat{u}_h) \]

- Global weak form:
  \[ a(\lambda_h, \mu) = b(\mu), \quad \forall \mu \in M^p_h \]
  \[ a(\lambda, \mu) = \sum_\kappa a_\kappa(\lambda_h, \mu) = \sum_\kappa - \left\langle (\hat{F}_h + \hat{G}_h)^{\lambda,0} \cdot n, \mu \right\rangle_{\partial \kappa} \]
  \[ b(\mu) = \sum_\kappa b_\kappa(\mu) = \sum_\kappa \left\langle (\hat{F}_h + \hat{G}_h)^{\lambda,0} \cdot n, \mu \right\rangle_{\partial \kappa} \]
Domain Decomposition

- **Local bilinear form:**
  - \( a_i(\lambda, \mu) = \sum_{\kappa \in \Omega_i} a_{\kappa}(\lambda, \mu) \)
  - \( b_i(\mu) = \sum_{\kappa \in \Omega_i} b_{\kappa}(\mu) \)

- **Local stiffness matrix and load vector:**
  - \( A^{(i)} = \begin{bmatrix} A_{II}^{(i)} & A_{I\Gamma}^{(i)} \\ A_{\Gamma I}^{(i)} & A_{\Gamma\Gamma}^{(i)} \end{bmatrix} \)
  - \( b^{(i)} = \begin{bmatrix} b_{I}^{(i)} \\ b_{\Gamma}^{(i)} \end{bmatrix} \)

- \( a_i(\lambda, \mu) = b_i(\mu) \) or \( A^{(i)} x^{(i)} = b^{(i)} \) correspond to local problems with Neumann interface condition on \( \Gamma \)

- **Global system formed by assembling with \( \Gamma \) DOFs**

\[
A = \sum_{i=1}^{N} R^{(i)^T} A^{(i)} R^{(i)} = \begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix}
\]

- \( R^{(i)} \) extracts global degrees of freedom associated with \( \Omega_i \)
BDDC preconditioner

- Interface DOFs are partitioned into dual and primal DOFs:
  \[ u^{(i)} = \begin{bmatrix} u^{(i)}_l & u^{(i)}_\Delta & u^{(i)}_\Pi \end{bmatrix}^T = \begin{bmatrix} u^{(i)}_r & u^{(i)}_\Pi \end{bmatrix}^T \]
  - Dual DOFs, \( u_\Delta \), correspond to functions with zero averages on subdomain interfaces
  - Primal DOFs, \( u_\Pi \), have non-zero averages on interfaces between subdomains

- Partially assembled system obtained by assembling only with respect to primal DOFs
  \[ \tilde{A} = \sum_{i=1}^N \tilde{R}^{(i)}^T A^{(i)} \tilde{R}^{(i)} = \begin{bmatrix} A_{rr} & A_{r\Pi} \\ A_{\Pi r} & A_{\Pi\Pi} \end{bmatrix} \]

- BDDC preconditioner:
  \[ M_{\text{BDDC}}^{-1} = \tilde{\mathcal{H}}_1 \tilde{A}^{-1} \tilde{\mathcal{H}}_2 \]
  - Lumped BDDC: \( \tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2 \) are simple averaging operators
  - BDDC: \( \tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2 \) averaging with harmonic extensions
BDDC preconditioner

- Solution of partially assembled problem:
  \[
  \tilde{A}^{-1} = \psi S_0 \psi^* T + \sum_{i=1}^{N} \tilde{R}(i)^T \tilde{A}(i)^{-1} \tilde{R}(i)
  \]

- \(\tilde{A}(i)^{-1}\) corresponds to solution of constrained Neumann problem:
  \[
  \begin{bmatrix}
  A(i) & B(i)^T \\
  B(i) & 0
  \end{bmatrix}
  \begin{bmatrix}
  \hat{A}(i)^{-1} r_i \\
  *
  \end{bmatrix}
  =
  \begin{bmatrix}
  r_i \\
  0
  \end{bmatrix}
  \]

- Coarse basis functions:
  \[
  \begin{bmatrix}
  A(i) & B(i)^T \\
  B(i) & 0
  \end{bmatrix}
  \begin{bmatrix}
  \psi_i \\
  *
  \end{bmatrix}
  =
  \begin{bmatrix}
  0 \\
  1
  \end{bmatrix}
  \]

- Coarse system matrix:
  \[
  S_o = \sum_{i=1}^{N} R_{\Pi}^{(i)^T} \psi_i^T A_i \psi_i R_{\Pi}^{(i)}
  \]

- \(B^{(i)}\)'s are constraints corresponding to primal DOFs
- \(R_{\Pi}^{(i)}\) extracts local primal DOFs from all primal DOFs
<table>
<thead>
<tr>
<th>Preconditioner</th>
<th>Scalable?</th>
<th>Condition number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive Schwarz (ASM)</td>
<td>no</td>
<td>$\kappa &lt; C \frac{1}{H^2} \left(1 + \left(p^2 \frac{H}{\delta}\right)\right)^2$</td>
</tr>
<tr>
<td>ASM with coarse space</td>
<td>yes</td>
<td>$\kappa &lt; C \left(1 + \left(p^2 \frac{H}{\delta}\right)\right)^2$</td>
</tr>
<tr>
<td>Lumped BDDC</td>
<td>yes</td>
<td>$\kappa &lt; C \left(1 + \left(p^2 \frac{H}{\delta}\right)\right)^2$</td>
</tr>
<tr>
<td>BDDC</td>
<td>yes</td>
<td>$\kappa &lt; C \left(1 + \log \left(p^2 \frac{H}{\delta}\right)\right)^2$</td>
</tr>
</tbody>
</table>

- Coarse space needed to correct low frequency error modes
- Preconditioners with coarse spaces are scalable
- Smooth extensions/averaging operators desired so as not to introduce high-frequency error modes from local solves
- Performance of BDDC preconditioner superior for larger subdomains as condition number/iteration count only weakly dependent on number of elements/subdomain
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Scalar Advection-Diffusion Equation

**Governing equation:**
- \( \frac{\partial u}{\partial t} + \vec{a} \cdot \nabla u - \nu \nabla^2 u = f \)
- Peclet number \( Pe = \frac{|\vec{a}|L}{\nu} \)
  - \( Pe << 1 \) - diffusion dominated
  - \( Pe >> 1 \) - advection dominated

**Sample Problem:**
- Uniform flow: \( \vec{a} = \begin{bmatrix} 1 & 0 \end{bmatrix} \)
- Exact solution: \( u = 1 - e^{-\frac{y}{\sqrt{\mu(x-x_0)}}} \)

Flow Solution | Uniform Mesh | Anisotropic Mesh
Scalar Advection-Diffusion: Diffusion Dominated

128 elem / proc, p = 1

128 elem / proc, p = 5

Uniform structured mesh problem with fixed $Pe_h = 1$

- Preconditioners with coarse spaces are scalable
- All preconditioners show dependence on solution order
Scalar Advection-Diffusion: Diffusion Dominated

128 elem / proc, p = 1

8192 elem / proc, p = 1

Uniform structured mesh problem with fixed $Pe_h = 1$

✔ Iteration count for BDDC preconditioner only weakly dependent on number of elements per subdomain
Scalar Advection-Diffusion: Advection Dominated

2D Advection-Diffusion problem with $Pe_h = 1000$

- Coarse space provides no additional benefit
- Iteration count proportional to number of subdomains in convective direction
Scalar Advection-Diffusion: Advection Dominated

2D Advection-Diffusion, $Pe_h = 1000$, $p = 1$, 128 elem /proc

- Coarse space provides no additional benefit
- Iteration count proportional to number of subdomains in convective direction
Scalar Advection-Diffusion High Pe Anisotropic Mesh

128 elem / proc, p = 1

8192 elem / proc, p = 1

2D Advection-Diffusion problem with $Pe = 10^6$, $p = 1$

- $Pe_{hy} \approx 1$ for well resolved solution
- Coarse space is necessary to ensure good performance
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Lineraized Navier-Stokes Equations:

\[
\frac{W}{\Delta t} + A_i \frac{\partial W}{\partial x_i} + K_{ij} \frac{\partial W}{\partial x_i \partial x_j} = F
\]

- \( A_i \), symmetric positive definite
- \( K \), symmetric positive semi-definite

Sample Problem:

\[
M = M_\infty \begin{bmatrix} 1 - e^{-\frac{y}{\sqrt{\mu}}} & 0 \\
1 & 0
\end{bmatrix}
\]
2D Linearized Euler, p = 1, 128 elem /proc

- Characteristic travel is multiple directions, reflecting off interfaces and boundaries
- Residual drops after a number of iterations proportional to the number of subdomains
Uniform structured mesh problem with fixed $Re_h = 1000$
Linearized Navier-Stokes Equations

128 elem / proc, p = 1

2048 elem / proc, p = 1

Uniform structured mesh problem with fixed $Re_h = 1$
Linearized Navier-Stokes Equations

128 elem / proc, p = 1

2048 elem / proc, p = 1

2D Linearized Navier-Stokes problem with $Re = 10^6$, $p = 1$
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Summary

 Contributions:
- Applied BDDC preconditioner for HDG discretizations
- Demonstrated the need for a coarse space correction when solving high Reynolds number flow on anisotropic meshes

 On going work:
- Generalized Robin-Robin interface condition for systems
- Application to fully non-linear Euler/Navier-Stokes problems
Questions?