Domain Decomposition Preconditioners for Higher-order Hybridizable Discontinuous Galerkin Discretizations

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Outline

1. Introduction and Motivation
2. Background
3. Scalar Advection-Diffusion Equation
4. Linearized Navier-Stokes Equations
5. Summary and Conclusions

Outline

Motivation: Developing Scalable CFD

Objective: Perform high-fidelity CFD simulations using high-performance computing in similar amount of time as typical industrial simulations on commodity hardware

- High performance computing in aerospace
  - Typical large jobs remain in O(100) processors
  - "The scalability of most of [CFD] codes tops out around 512 cpus ..." – Mavriplis et al [2007]
  - Algorithmic improvements are needed to scale to > 10,000 cpus
- Domain decomposition for elliptic problem
  - Independent local solvers on each processor
  - Coupled global solver
  - Small number of globally coupled DOF
  - Algorithms have been developed and used on > 100,000 cpus

Can we use domain decomposition theory for elliptic problems to develop better solvers for Euler/Navier-Stokes problems?

Approach: HDG/BDDC

- Hybridizable discontinuous Galerkin (HDG) discretization
  - Solution represented by piecewise polynomials on each element
  - Implicit
  - Higher-order solutions
  - Unstructured meshes
  - Well suited to hp-adaptation
- Balancing Domain Decomposition by Constraints (BDDC)
  - Preconditioner for the Schur complement problem
  - Developed by Dohrmann [2003] for structural mechanics applications
  - Makes use of the finite element assembly of the system matrix
  - Condition number bound $\kappa < C(1 + \log(\frac{H}{h}))^2$ for second order elliptic problems
  - ✔ Coarse space is defined algebraically using discrete harmonic functions

Outline

HDG Discretizations

- Governing equation:

$$\nabla \cdot (F(u) + G(u, \nabla u)) = f$$

  - u: state, $F(u)$: inviscid flux, $G(u)$: viscous flux

- Local Solver:

$$\begin{aligned} (\boldsymbol{q}_h, \mathbf{v})_\kappa + (u_h, \nabla \cdot \mathbf{v})_\kappa - \langle \hat{u}_h, \mathbf{v} \cdot \boldsymbol{n} \rangle_{\partial\kappa} &= 0, & \forall \mathbf{v} \in \mathbf{P}^p(\kappa) \\ -(F + G, \nabla w)_\kappa + \left\langle (\hat{F}_h + \hat{G}_h) \cdot \boldsymbol{n}, w \right\rangle_{\partial\kappa} &= (f, w)_\kappa & \forall w \in P^p(\kappa) \end{aligned}$$

- Numerical Fluxes:

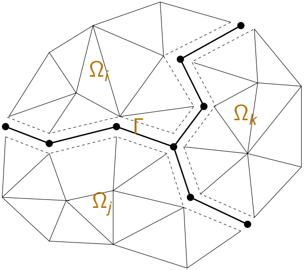
$$(\hat{F}_h + \hat{G}_h) \cdot \boldsymbol{n} = (F(\hat{u}_h) + G(\hat{u}_h, \boldsymbol{q}_h)) \cdot \boldsymbol{n} + S(\hat{u}_h)(u_h - \hat{u}_h)$$

- Global weak form:

$$a(\lambda_h, \mu) = b(\mu) \qquad \forall \mu \in M_h^p$$

  - $a(\lambda, \mu) = \sum_\kappa a_\kappa(\lambda_h, \mu) = \sum_\kappa -\left\langle (\hat{F}_h + \hat{G}_h)^{\lambda,0} \cdot \boldsymbol{n}, \mu \right\rangle_{\partial\kappa}$
  - $b(\mu) = \sum_\kappa b_\kappa(\mu) = \sum_\kappa \left\langle (\hat{F}_h + \hat{G}_h)^{\lambda,0} \cdot \boldsymbol{n}, \mu \right\rangle_{\partial\kappa}$

Domain Decomposition



- Local bilinear form:
  - $a_i(\lambda,\mu) = \sum_{\kappa\in\Omega_i} a_\kappa(\lambda,\mu)$
  - $b_i(\mu) = \sum_{\kappa\in\Omega_i} b_\kappa(\mu)$
- Local stiffness matrix and load vector:
  - $A^{(i)} = \begin{bmatrix} A^{(i)}_{II} & A^{(i)}_{I\Gamma} \\ A^{(i)}_{\Gamma I} & A^{(i)}_{\Gamma\Gamma} \end{bmatrix}, \quad b^{(i)} = \begin{bmatrix} b^{(i)}_I \\ b^{(i)}_\Gamma \end{bmatrix}$

- $a_i(\lambda,\mu) = b_i(\mu)$ or $A^{(i)}x^{(i)} = b^{(i)}$ correspond to local problems with Neumann interface condition on Γ
- Global system formed by assembling with Γ DOFs

$$A = \sum_{i=1}^{N} R^{(i)^T} A^{(i)} R^{(i)} = \begin{bmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{bmatrix}$$

  - $R^{(i)}$ extracts global degrees of freedom associated with Ω_i

BDDC preconditioner

- Interface DOFs are partitioned into dual and primal DOFs:

$$u^{(i)} = \begin{bmatrix} u_I^{(i)} & u_\Delta^{(i)} & u_\Pi^{(i)} \end{bmatrix}^T = \begin{bmatrix} u_r^{(i)} & u_\Pi^{(i)} \end{bmatrix}^T$$

  - Dual DOFs, u_Δ, correspond to functions with zero averages on subdomain interfaces
  - Primal DOFs, u_Π, have non-zero averages on interfaces between subdomains
- Partially assembled system obtained by assembling only with respect to primal DOFs

$$\tilde{A} = \sum_{i=1}^{N} \tilde{R}^{(i)^T} A^{(i)} \tilde{R}^{(i)} = \begin{bmatrix} A_{rr} & A_{r\Pi} \\ A_{\Pi r} & A_{\Pi\Pi} \end{bmatrix}$$

- BDDC preconditioner:

$$M_{\text{BDDC}}^{-1} = \tilde{\mathcal{H}}_1 \tilde{A}^{-1} \tilde{\mathcal{H}}_2$$

  - Lumped BDDC: $\tilde{\mathcal{H}}_1$, $\tilde{\mathcal{H}}_2$ are simple averaging operators
  - BDDC: $\tilde{\mathcal{H}}_1$, $\tilde{\mathcal{H}}_2$ averaging with harmonic extensions

BDDC preconditioner

- Solution of partially assembled problem:

$$\tilde{A}^{-1} \;=\; \Psi S_o \Psi^{*T} + \sum_{i=1}^{N} \tilde{R}^{(i)T} \tilde{A}^{(i)-1} \tilde{R}^{(i)}$$

- $\tilde{A}^{(i)-1}$ corresponds to solution of constrained Neumann problem:

$$\begin{bmatrix} A^{(i)} & B^{(i)T} \\ B^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \hat{A}^{(i)-1} r_i \\ * \end{bmatrix} = \begin{bmatrix} r_i \\ 0 \end{bmatrix}$$

- Coarse basis functions:

$$\begin{bmatrix} A^{(i)} & B^{(i)T} \\ B^{(i)} & 0 \end{bmatrix} \begin{bmatrix} \Psi_i \\ * \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

- Coarse system matrix:

$$S_o = \sum_{i=1}^{N} R_{\Pi}^{(i)T} \Psi_i^{*T} A_i \Psi_i R_{\Pi}^{(i)}$$

  - $B^{(i)}$'s are constraints corresponding to primal DOFs
  - $R_{\Pi}^{(i)}$ extracts local primal DOFs from all primal DOFs

Domain Decomposition Theory: Elliptic Problems

Preconditioner	Scalable?	Condition number
Additive Schwarz (ASM)	no	$\kappa < C \frac{1}{H^2} \left(1 + \left(p^2 \frac{H}{\delta}\right)\right)^2$
ASM with coarse space	yes	$\kappa < C \left(1 + \left(p^2 \frac{H}{\delta}\right)\right)^2$
Lumped BDDC	yes	$\kappa < C \left(1 + \left(p^2 \frac{H}{h}\right)\right)^2$
BDDC	yes	$\kappa < C \left(1 + \log \left(p^2 \frac{H}{h}\right)\right)^2$

- Coarse space needed to correct low frequency error modes
- ✔ Preconditioners with coarse spaces are scalable
- Smooth extensions/averaging operators desired so as not to introduce high-frequency error modes from local solves
- ✔ Performance of BDDC preconditioner superior for larger subdomains as condition number/iteration count only weakly dependent on number of elements/ subdomain

Outline

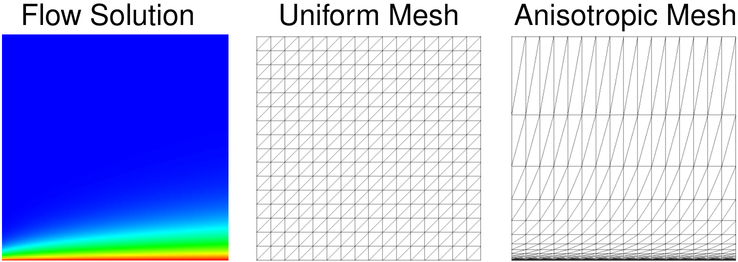
Scalar Advection-Diffusion Equation

Governing equation:

- $\frac{\partial u}{\partial t} + \vec{a} \cdot \nabla u - \nu \nabla^2 u = f$
- Peclet number $Pe = \frac{|\vec{a}|L}{\nu}$
  - $Pe << 1$ - diffusion dominated
  - $Pe >> 1$ - advection dominated

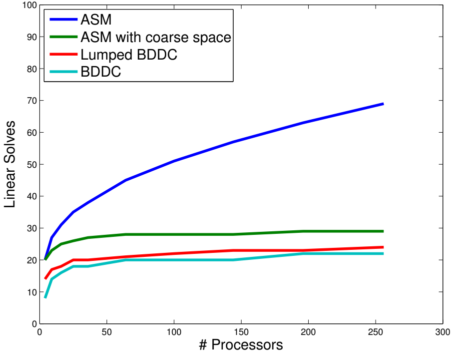
Sample Problem:

- Uniform flow: $\vec{a} = \begin{bmatrix} 1 & 0 \end{bmatrix}$
- Exact solution: $u = 1 - e^{-\frac{y}{\sqrt{\mu(x-x_0)}}}$

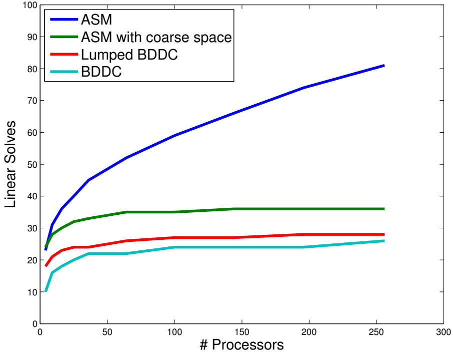


Scalar Advection-Diffusion: Diffusion Dominated

128 elem / proc, p = 1



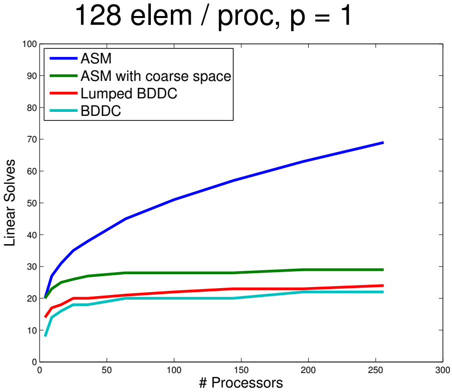
128 elem / proc, p = 5

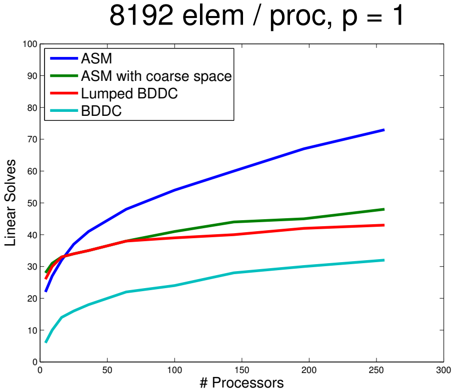


Uniform structured mesh problem with fixed $Pe_h = 1$

- ✔ Preconditioners with coarse spaces are scalable
- All preconditioners show dependence on solution order

Scalar Advection-Diffusion: Diffusion Dominated

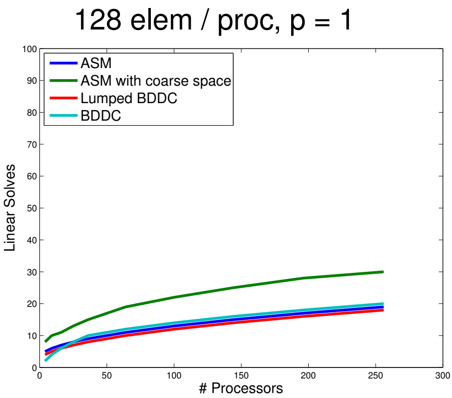


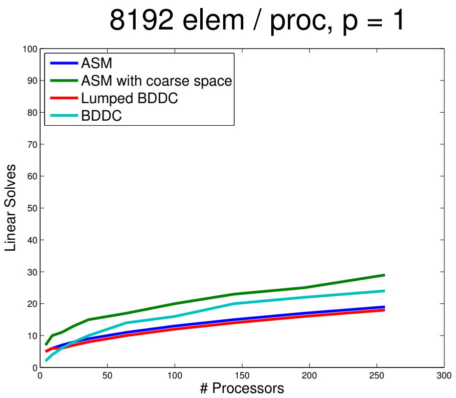


Uniform structured mesh problem with fixed $Pe_h = 1$

✔ Iteration count for BDDC preconditioner only weakly dependent on number of elements per subdomain

Scalar Advection-Diffusion: Advection Dominated

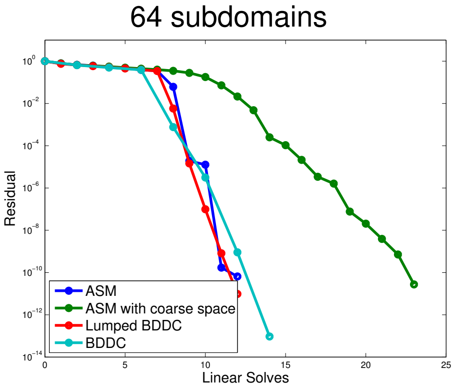


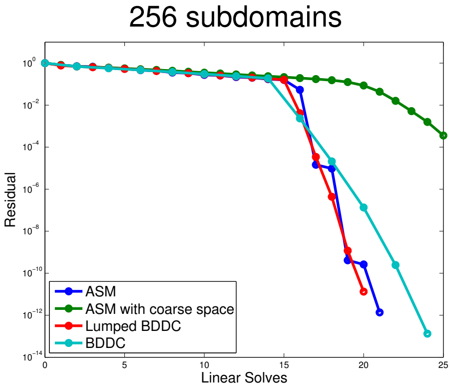


2D Advection-Diffusion problem with $Pe_h = 1000$

- Coarse space provides no additional benefit
- Iteration count proportional to number of subdomains in convective direction

Scalar Advection-Diffusion: Advection Dominated



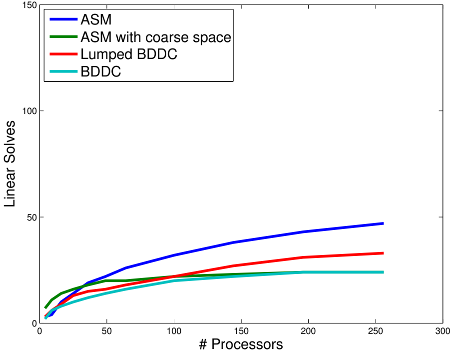


2D Advection-Diffusion, $Pe_h = 1000$, p = 1, 128 elem /proc

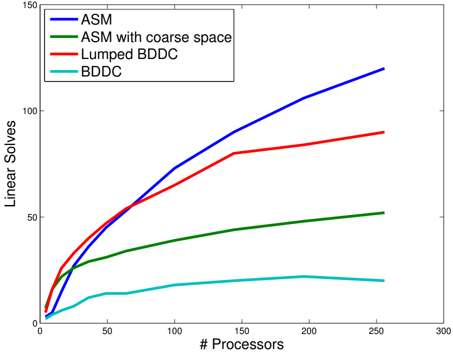
- Coarse space provides no additional benefit
- Iteration count proportional to number of subdomains in convective direction

Scalar Advection-Diffusion High Pe Anisotropic Mesh

128 elem / proc, p = 1



8192 elem / proc, p = 1



2D Advection-Diffusion problem with $Pe = 10^6$, $p = 1$

- $Pe_{h_y} \approx 1$ for well resolved solution
- Coarse space is necessary to ensure good performance

Outline

Linearized Navier-Stokes Equations

- Linearized Navier-Stokes Equations:

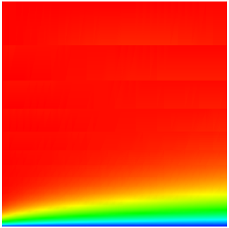
$$\frac{W}{\Delta t} + A_i \frac{\partial W}{\partial x_i} + K_{ij} \frac{\partial W}{\partial x_i \partial x_j} \quad = \quad F$$

  - A_i, symmetric positive definite
  - K, symmetric positive semi-definite

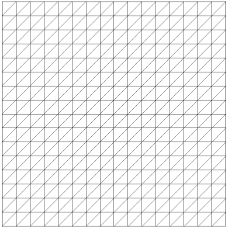
Sample Problem:

- $M = M_\infty \left[\begin{array}{cc} 1 - e^{-\frac{y}{\sqrt{\mu}}} & 0 \end{array} \right]$

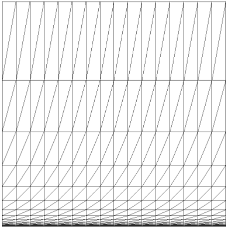
Entropy



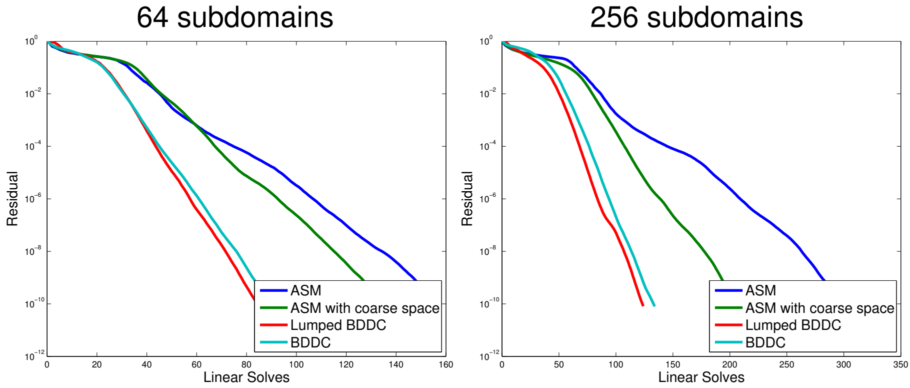
Uniform Mesh



Anisotropic Mesh



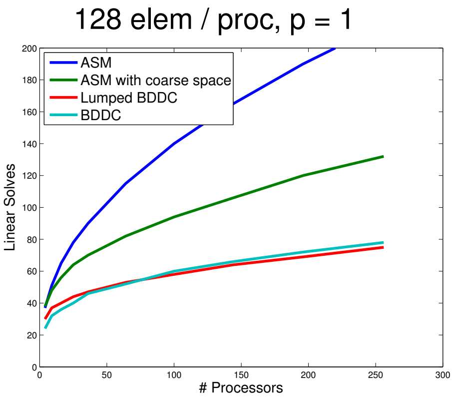
Linearized Euler Equations

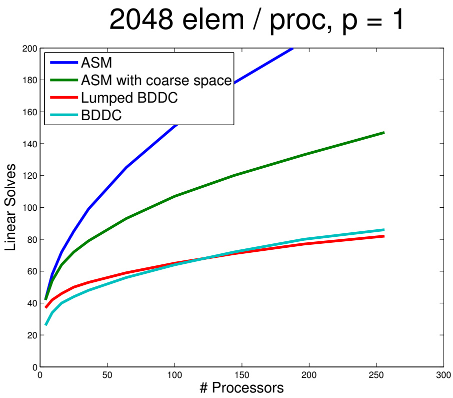


2D Linearized Euler, p = 1, 128 elem /proc

- Characteristic travel is multiple directions, reflecting off interfaces and boundaries
- Residual drops after a number of iterations proportional to the number of subdomains

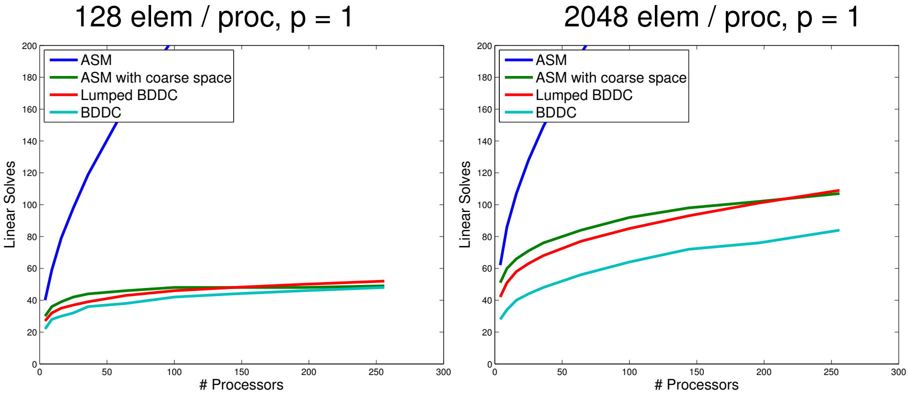
Linearized Navier-Stokes Equations





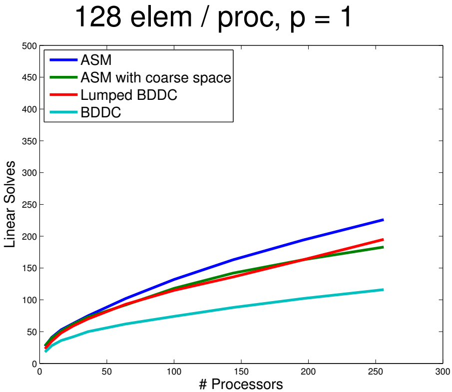
Uniform structured mesh problem with fixed $Re_h = 1000$

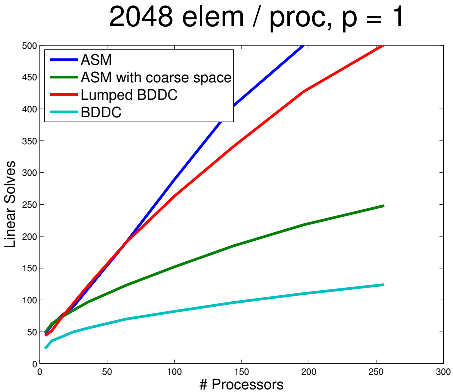
Linearized Navier-Stokes Equations



Uniform structured mesh problem with fixed $Re_h = 1$

Linearized Navier-Stokes Equations





2D Linearized Navier-Stokes problem with $Re = 10^6$, $p = 1$

Outline

Summary

- **Contributions**:
  - Applied BDDC preconditioner for HDG discretizations
  - Demonstrated the need for a coarse space correction when solving high Reynolds number flow on anisotropic meshes
- **On going work**:
  - Generalized Robin-Robin interface condition for systems
  - Application to fully non-linear Euler/Navier-Stokes problems

Questions?