ANALYSIS OF DUAL CONSISTENCY FOR DISCONTINUOUS GALERKIN DISCRETIZATIONS OF SOURCE TERMS *

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Abstract. The effects of dual consistency on discontinuous Galerkin (DG) discretizations of solution and solution gradient dependent source terms are examined. Two common discretizations are analyzed: the standard weighting technique for source terms and the mixed formulation. It is shown that if the source term depends on the first derivative of the solution, the standard weighting technique leads to a dual inconsistent scheme. A straightforward procedure for correcting this dual inconsistency and arriving at a dual consistent discretization is demonstrated. The mixed formulation, where the solution gradient in the source term is replaced by an additional variable that is solved for simultaneously with the state, leads to an asymptotically dual consistent discretization. Numerical results for a one-dimensional test problem confirm that the dual consistent and asymptotically dual consistent schemes achieve higher asymptotic convergence rates with grid refinement than a similar dual inconsistent scheme for both the primal and adjoint solutions as well as a simple functional output.

Key words. discontinuous Galerkin, dual consistency, high-order

AMS subject classifications. 65N30, 65N15

1. Introduction.

In recent years, the discontinuous Galerkin (DG) finite element method has become a popular tool in the numerical simulation of many complex physical phenomena. In particular, many researchers have investigated high-order accurate DG discretizations of the Euler and Navier-Stokes equations for use in computational fluid dynamics [22, 11, 5, 7, 6, 12, 4]. In this context, DG is attractive because it allows the development of high-order accurate discretizations with element-wise compact stencils. These compact stencils simplify the task of achieving high-order accuracy for problems involving complex geometries, where unstructured meshes are often employed, and allow the development of efficient solution methods.

In this paper, high-order accurate DG discretizations of source terms depending on the state and its gradient are examined. Interest in such terms stems from the Reynolds-averaged Navier-Stokes (RANS) equations and, specifically, from the turbulence models used to close the RANS equations. For example, the Spalart-Allmaras turbulence model [25] incorporates state and state derivative dependent source terms to model the production, destruction, and diffusion of turbulent eddy viscosity, and state derivative dependent source terms appear in both the $k-\epsilon$ and $k-\omega$ turbulence models [28].

The focus of the paper is the impact of dual consistency on source term discretizations. Dual consistency provides a connection between the continuous and discrete dual problems. In particular, if a discretization is dual consistent, then the exact solution of the strong form of the dual problem satisfies the discrete dual problem taken about the exact solution of the strong form of the primal problem. A more precise definition of dual consistency is given in Section 2.

For many types of discretization, algorithms involving the dual problem have

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become popular for design optimization, error estimation, and grid adaptation [21, 1, 14, 16, 8, 9, 17, 27]. It is well known that dual consistency can significantly impact the performance of these algorithms. For example, Collis and Heinkenschloss [13] showed that when applying a dual inconsistent streamline upwind/Petrov-Galerkin (SUPG) method for linear advection-diffusion to an optimal control problem, superior results are obtained using a direct discretization of the continuous dual problem as opposed to the discrete dual problem derived from the primal discretization. Specifically, both the control function and the adjoint solution converge at a higher rate when the continuous dual problem is discretized directly.

For DG discretizations, Harriman *et al.* [19, 18] examined symmetric and non-symmetric interior penalty (SIPG and NIPG, respectively) DG methods for the solution of Poisson's equation. They showed that to achieve optimal convergence rates for a linear functional output, the dual consistent method (i.e. SIPG) must be used. Lu [23] considered the impact of dual consistency on the accuracy of functional outputs computed using DG discretizations of the Euler and Navier-Stokes equations. He demonstrated the importance of implementing the boundary conditions on the primal problem in a dual consistent manner. In particular, when using dual consistent boundary conditions, super-convergent functional output results were obtained, while, when using a dual inconsistent boundary condition treatment, significant degradation of the output convergence rates was observed. More recently, Hartmann [20] proposed a framework for analyzing the dual consistency of DG discretizations. He uses the framework to expand upon the analysis of the SIPG discretization for the Navier-Stokes equations, proposing a modification of the boundary conditions to make the scheme dual consistent. Similar to the results shown by Lu, Hartmann's modification of the SIPG scheme produces superior results to the original, dual inconsistent boundary condition treatment.

Furthermore, it is well known that dual consistency can impact the convergence of the L^2 norm of the error in the primal solution [3]. For example, for many DG discretizations of Poisson's equation, standard proofs of order of accuracy of the solution error in the L^2 norm exist. Typically these proofs rely on the Aubin-Nitsche "duality trick" [26, 24] to obtain an optimal estimate in the L^2 norm given an optimal estimate in the energy norm [10, 3]. The use of this duality argument requires that the scheme be dual consistent. Thus, some dual inconsistent methods—e.g. NIPG and the Baumann-Oden method—do not achieve optimal accuracy in the L^2 norm, and dual inconsistent methods that do achieve optimal accuracy in the L^2 norm are typically super-penalized [3].

The paper begins with a brief review of the definition of dual consistency in Section 2. The standard weighting DG discretization of source terms is considered in Section 3. It is shown that, while this treatment of solution derivative dependent source terms leads to a dual inconsistent DG discretization, dual consistency can be achieved by adding terms proportional to the jumps in the solution between elements to the discretization. Mixed formulations for the source term are analyzed in Section 4. The resulting discretizations are shown to be asymptotically dual consistent. Finally, numerical results for a simple test problem are shown in Section 5.

2. Dual Consistency Definition and Preliminaries. Consider the following primal problem: compute $\mathcal{J}(u)$, where $\mathcal{J} : \mathcal{V} \to \mathbb{R}$ is a functional of interest, $\mathcal{V}$ is an appropriate function space, and $u \in \mathcal{V}$ solves

$$R(u, v) = 0 \quad \forall v \in \mathcal{V},$$

where $R : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is the weak form of a PDE of interest. For simplicity, it is assumed that this primal problem and its dual are well-posed.

Let $\mathcal{V}_h^p$ be a finite dimensional vector space of piecewise polynomial functions of degree at most p on a triangulation, T_h, of the domain of interest, $\Omega \subset \mathbb{R}^n$, into elements, κ, such that $\bar{\Omega} = \cup_{\kappa \in T_h} \bar{\kappa}$. In particular,

$$\mathcal{V}_h^p \equiv \{v \in L^2(\Omega) \,|\, v|_\kappa \in P^p,\, \forall \kappa \in T_h\},$$

where P^p denotes the space of polynomial functions of degree at most p.

Let $\mathcal{W}_h^p \equiv \mathcal{V}_h^p + \mathcal{V}$, where

$$\mathcal{V}_h^p + \mathcal{V} \equiv \{h = f + g \mid f \in \mathcal{V}_h^p,\, g \in \mathcal{V}\}.$$

Then, consider a general DG discretization of the primal problem: find $u_h \in \mathcal{V}_h^p$ such that

$$R_h(u_h, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p,$$

where $R_h : \mathcal{W}_h^p \times \mathcal{W}_h^p \to \mathbb{R}$ is a semi-linear form derived from the weak form of the primal problem. For the remainder of the paper, it is assumed that this discrete problem is also well-posed.

Let $\mathcal{J}_h : \mathcal{W}_h^p \to \mathbb{R}$ be the discrete functional of interest. Then, the discrete dual problem is given by the following statement: find $\psi_h \in \mathcal{V}_h^p$ such that

$$R_h'[u_h](v_h, \psi_h) = \mathcal{J}_h'[u_h](v_h), \quad \forall v_h \in \mathcal{V}_h^p$$

where $R_h'[u_h](\cdot, \psi_h) : \mathcal{W}_h^p \to \mathbb{R}$ is the linear functional given by evaluating the Frechét derivative of the function $N_{\psi_h} : \mathcal{W}_h^p \to \mathbb{R}$ at u_h, where, for fixed $\psi_h \in \mathcal{W}_h^p$,

$$N_{\psi_h}(w_h) = R_h(w_h, \psi_h), \quad \forall w_h \in \mathcal{W}_h^p.$$

Similarly, $\mathcal{J}_h'[u_h](\cdot) : \mathcal{W}_h^p \to \mathbb{R}$ is the linear functional given by evaluating the Frechét derivative of $\mathcal{J}_h$ at u_h.

Two concepts of dual consistency are used in this work: dual consistency, defined in Definition 2.1, and asymptotic dual consistency, defined in Definition 2.2. Dual consistency has been used in the analysis of DG methods by multiple authors. For example, [19, 18, 3] define dual consistency for linear problems. A general definition of dual consistency for nonlinear problems is provided by both Lu [23] and Hartmann [20]. Lu also defines asymptotic dual consistency for general nonlinear problems. The definitions used here follow [23].

Definition 2.1. *The discretization defined by the semi-linear form, R_h, and discrete functional, $\mathcal{J}_h$, is said to be* dual consistent *if, given exact solutions $u \in \mathcal{V}$ and $\psi \in \mathcal{V}$ of the continuous primal and dual problems, respectively,*

$$R_h'[u](v, \psi) = \mathcal{J}_h'[u](v), \quad \forall v \in \mathcal{W}_h^p.$$

Definition 2.2. *The discretization defined by the semi-linear form, R_h, and discrete functional, $\mathcal{J}_h$, is said to be* asymptotically dual consistent *if, given exact solutions $u \in \mathcal{V}$ and $\psi \in \mathcal{V}$ of the continuous primal and dual problems, respectively,*

$$\lim_{h \to 0} \left(\sup_{v \in \mathcal{W}_h^p, \|v\|_{\mathcal{W}_h^p} = 1} |R_h'[u](v, \psi) - \mathcal{J}_h'[u](v)| \right) = 0.$$

Clearly, all dual consistent discretizations are automatically asymptotically dual consistent. In this work, a discretization will be referred to as asymptotically dual consistent only if it is not also dual consistent.

3. The Standard Weighting Technique for Source Terms. This section considers DG discretizations of source terms depending on the state and first derivatives of the state. As will be shown, the simple approach of weighting by a test function and integrating leads to a dual inconsistent scheme for source terms that depend on derivatives of the state. However, a dual consistent discretization can be constructed by adding terms to the discretization.

Let $u \in \mathcal{V} \equiv H^1(\Omega)$ be the solution of the following scalar problem:

$$-\nabla \cdot (\nu \nabla u) = f(u, \nabla u) \quad \text{for } x \in \Omega \subset \mathbb{R}^n \qquad (3.1)$$
$$u = 0 \quad \text{for } x \in \partial\Omega,$$

where $f \in \mathcal{C}^1(\mathbb{R}^{n+1})$ is the source term of interest. As before, it is assumed that this problem and its dual are well-posed.

Let $\mathcal{J} : H^1(\Omega) \to \mathbb{R}$ be a functional of interest defined as

$$\mathcal{J}(w) = \int_\Omega J(w), \qquad (3.2)$$

where $J \in \mathcal{C}(\mathbb{R})$. For simplicity, the functional chosen here does not include boundary integrals. The inclusion of boundary integrals would slightly alter the boundary conditions on the dual problem and the analysis of the boundary terms shown here. However, these modifications are not central to the analysis of the source term discretization. For an analysis of dual consistency for equations without source terms that includes functionals with boundary integrals, see [23].

For $\mathcal{J}$, the dual problem is given by

$$-\nabla \cdot (\nu \nabla \psi) - D_1 f(u, \nabla u)\psi + \nabla \cdot (D_{\nabla u} f(u, \nabla u)\psi) = J'[u] \quad \text{for } x \in \Omega, \qquad (3.3)$$

where ψ is the adjoint state, $D_1 f(u, \nabla u)$ is the partial derivative of f with respect to u evaluated at $(u, \nabla u)$ and

$$D_{\nabla u} f(u, \nabla u) = [D_2 f(u, \nabla u), \ldots, D_{n+1} f(u, \nabla u)]^T$$

where $D_i f(u, \nabla u)$ is the partial derivative of f with respect to $\frac{\partial u}{\partial x_{i-1}}$ for $2 \le i \le n+1$, evaluated at $(u, \nabla u)$. The boundary conditions on the dual problem can be written in the following weak form:

$$-\int_{\partial\Omega} \psi q = 0, \quad \forall q \in H^{-1/2}(\partial\Omega).$$

Consider the following DG discretization: find $u_h \in \mathcal{V}_h^p$ such that

$$R_h(u_h, v_h) \equiv B_h(u_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(u_h, \nabla u_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p, \qquad (3.4)$$

where B_h is a consistent and dual consistent bilinear form for the diffusion operator (e.g. BR2 [6, 4] or LDG [11, 12]). Furthermore, define the discrete functional of interest, $\mathcal{J}_h : \mathcal{W}_h^p \to \mathbb{R}$, as

$$\mathcal{J}_h(w_h) = \sum_{\kappa \in T_h} \int_\kappa J(w_h). \qquad (3.5)$$

In the following dual consistency analysis, additional smoothness is assumed for the exact primal and dual solutions, u and ψ. In particular, it is assumed that $u, \psi \in H^2(\Omega)$. This smoothness assumption is common in the analysis of dual consistency for discretizations of second-order operators [3, 23].

PROPOSITION 3.1. *The discretization defined by the semi-linear form R_h, defined in* (3.4)*, together with the discrete functional $\mathcal{J}_h$, defined in* (3.5)*, is dual inconsistent.*

Proof. Linearizing R_h about the exact solution and integrating by parts gives

$$\begin{aligned} R_h'[u](w_h, v_h) = B_h(w_h, v_h) &- \sum_{\kappa \in T_h} \int_\kappa w_h (D_1 f(u, \nabla u) v_h - \nabla \cdot (D_{\nabla u} f(u, \nabla u) v_h)) \\ &- \int_{\Gamma_i} ([\![w_h]\!] \cdot \{D_{\nabla u} f(u, \nabla u) v_h\} + \{w_h\} [\![D_{\nabla u} f(u, \nabla u) v_h]\!]) \\ &- \int_{\partial\Omega} w_h (D_{\nabla u} f(u, \nabla u) v_h) \cdot \vec{n}, \end{aligned}$$

where $[\![\cdot]\!]$ and $\{\cdot\}$ denote the standard jump and average operators, respectively, on interior faces (see e.g. [3]), $\vec{n}$ is the outward pointing unit normal vector, and Γ_i denotes the union of the interior faces of the triangulation T_h.

The assumptions $\psi \in H^2(\Omega)$, $u \in H^2(\Omega)$, and $f \in \mathcal{C}^1(\mathbb{R}^{n+1})$, imply that $\{D_{\nabla u} f(u, \nabla u)\psi\} = D_{\nabla u} f(u, \nabla u)\psi$ and $[\![D_{\nabla u} f(u, \nabla u)\psi]\!] = 0$. Thus,

$$\begin{aligned} R_h'[u](v_h, \psi) = B_h(v_h, \psi) &- \sum_{\kappa \in T_h} \int_\kappa v_h (D_1 f(u, \nabla u)\psi - \nabla \cdot (D_{\nabla u} f(u, \nabla u)\psi)) \\ &- \int_{\Gamma_i} [\![v_h]\!] \cdot (D_{\nabla u} f(u, \nabla u)\psi) - \int_{\partial\Omega} v_h (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}. \end{aligned}$$

Evaluating the dual consistency using the discrete functional as defined in (3.5) gives

$$R_h'[u](v_h, \psi) - \mathcal{J}_h'[u](v_h) = (\mathcal{L}_{h,I}(u, \psi))(v_h) + (\mathcal{L}_{h,B}(u, \psi))(v_h),$$

where

$$(\mathcal{L}_{h,I}(u, \psi))(v_h) \equiv - \int_{\Gamma_i} [\![v_h]\!] \cdot (D_{\nabla u} f(u, \nabla u)\psi), \tag{3.6}$$

$$(\mathcal{L}_{h,B}(u, \psi))(v_h) \equiv - \int_{\partial\Omega} v_h (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}. \tag{3.7}$$

In general, there exists $v_h \in \mathcal{V}_h^p$ such that at least $(\mathcal{L}_{h,I}(u, \psi))(v_h)$ does not vanish. Thus, the scheme is dual inconsistent. Due to the boundary condition on the dual problem, the boundary term, $(\mathcal{L}_{h,B}(u, \psi))(v_h)$, will vanish if $(D_{\nabla u} f(u, \nabla u) v_h \cdot \vec{n})|_{\partial\Omega} \in H^{-1/2}(\partial\Omega)$. However, if this condition does not hold, the boundary term will also contribute to the dual inconsistency. ∎

Remark 3.1. It is possible to construct a dual consistent discretization by adding terms to the semi-linear form R_h. In particular, define a new bilinear form,

$$R_{h,DC}(w_h, v_h) \equiv R_h(w_h, v_h) + A_{h,I}(w_h, v_h) + A_{h,B}(w_h, v_h),$$

where $A_{h,I}$ will serve to eliminate the interior face dual inconsistency term, $\mathcal{L}_I$, and $A_{h,B}$ will serve to eliminate the boundary face dual inconsistency, $\mathcal{L}_B$. Furthermore, to maintain consistency, $A_{h,I}$ and $A_{h,B}$ must vanish when evaluated at u:

$$A_{h,I}(u, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p,$$
$$A_{h,B}(u, v_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p.$$

The interior face and boundary face contributions to the dual inconsistency are examined separately. To eliminate the dual inconsistency from the interior faces, the following term is added to the semi-linear form R_h:

$$A_{h,I}(w_h, v_h) = \int_{\Gamma_i} [\![w_h]\!] \cdot \{\vec{\beta}_i(w_h, v_h)\},$$

where dual consistency requires that $\{\vec{\beta}_i(u, \psi)\} = D_{\nabla u} f(u, \nabla u)\psi$.

To eliminate the boundary dual inconsistency, the following term is added to R_h:

$$A_{h,B}(w_h, v_h) = \int_{\partial\Omega} w_h \vec{\beta}_b(w_h, v_h) \cdot \vec{n},$$

where dual consistency requires $\vec{\beta}_b(u, \psi) = D_{\nabla u} f(u, \nabla u)\psi$.

PROPOSITION 3.2. *Let B_h be a dual consistent bilinear form corresponding to the diffusion operator. Let $\vec{\beta}_i$ be such that $\vec{\beta}_i(w, v) = D_{\nabla u} f(w, \nabla w)v$ for all $w, v \in H^2(\Omega)$. Let $\vec{\beta}_b$ be such that $\vec{\beta}_b(w, v) = D_{\nabla u} f(w, \nabla w)v$ for all $w, v \in H^2(\Omega)$. Then, the semi-linear form given by*

$$R_{h,DC}(w_h, v_h) = B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(w_h, \nabla w_h) + \int_{\Gamma_i} [\![w_h]\!] \cdot \{\vec{\beta}_i(w_h, v_h)\} + \int_{\partial\Omega} w_h \vec{\beta}_b(w_h, v_h) \cdot \vec{n},$$

together with the discrete functional $\mathcal{J}_h$, defined in (3.5)*, is dual consistent.*

Proof. Linearizing $R_{h,DC}$ gives

$$R'_{h,DC}[u](v_h, \psi) = R'_h[u](v_h, \psi) + \int_{\Gamma_i} [\![v_h]\!] \cdot (D_{\nabla u} f(u, \nabla u)\psi) + \int_{\partial\Omega} v_h (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}, \quad \forall v_h \in \mathcal{W}_h^p.$$

Thus,

$$R'_{h,DC}[u](v_h, \psi) - \mathcal{J}_h[u](v_h) = 0, \quad \forall v_h \in \mathcal{W}_h^p.$$

□

Remark 3.2. The choices of $\vec{\beta}_i$ and $\vec{\beta}_b$ are not fully determined by requiring dual consistency. One valid choice is given by

$$\vec{\beta}_i(w_h, v_h) = \{D_{\nabla u} f(w_h, \nabla w_h) v_h\}; \ \vec{\beta}_b(w_h, v_h) = D_{\nabla u} f(w_h, \nabla w_h) v_h.$$

Then,

$$R_{h,DC}(w_h, v_h) = B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f + \int_{\Gamma_i} [\![w_h]\!] \cdot \{D_{\nabla u} f(w_h, \nabla w_h) v_h\} + \int_{\partial\Omega} w_h (D_{\nabla u} f(w_h, \nabla w_h) v_h) \cdot \vec{n}. \tag{3.8}$$

However, if necessary, one could construct more complex functions that satisfy the dual consistency requirement as well as add stability to the discretization. For example, given that the dual problem is a convection-diffusion-reaction problem, one option to add stability is to upwind the dual consistency terms instead of averaging.

Remark 3.3. In addition to being dual consistent, if B_h is a consistent bilinear form for the diffusion operator, the discretization of Proposition 3.2 is *consistent* for any choice of $\vec{\beta}_i$ and $\vec{\beta}_b$ because, for the exact solution u, $[\![u]\!] = 0$ and $u|_{\partial\Omega} = 0$. Thus,

$$R_{h,DC}(u, v_h) = B_h(u, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(u, \nabla u) = 0, \quad \forall v_h \in \mathcal{V}_h^p.$$

4. The Mixed Formulation for Source Terms. In addition to the standard weighting source term treatment discussed in Section 3, another source term treatment of interest has appeared in the DG literature. In this method, known as the mixed formulation, the gradient of the state is replaced by a variable that is solved for simultaneously with the primal state [4]. Variants of this technique are widely used in DG discretizations of second-order operators. See [3] for a full analysis of those discretizations. This section provides a brief derivation of the mixed method as applied to source terms involving the gradient of the state. Furthermore, analysis of the mixed formulation shows that, in general, it is asymptotically dual consistent.

4.1. Discretization Derivation. Consider (3.1)—i.e. the model problem considered in Section 3—and consider the following discretization: find $u_h \in \mathcal{V}_h^p$ and $\vec{g}_h \in [\mathcal{V}_h^p]^n$ such that

$$R_h(u_h, v_h) \equiv B_h(u_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(u_h, \vec{g}_h) = 0, \quad \forall v_h \in \mathcal{V}_h^p, \tag{4.1}$$

$$\sum_{\kappa \in T_h} \int_\kappa \vec{\tau}_h \cdot \vec{g}_h = - \sum_{\kappa \in T_h} \int_\kappa u_h \nabla \cdot \vec{\tau}_h + \int_{\Gamma_i} \left([\![\hat{u}(u_h)]\!] \cdot \{\vec{\tau}_h\} + \{\hat{u}(u_h)\}[\![\vec{\tau}_h]\!] \right) + \int_{\partial\Omega} u^b(u_h) \vec{\tau}_h \cdot \vec{n}, \quad \forall \vec{\tau}_h \in [\mathcal{V}_h^p]^n \tag{4.2}$$

where $\hat{u}$ and u^b are numerical flux functions on interior and boundary faces respectively. Integrating by parts on (4.2) gives

$$\sum_{\kappa \in T_h} \int_\kappa \vec{\tau}_h \cdot \vec{g}_h = \sum_{\kappa \in T_h} \int_\kappa \vec{\tau}_h \cdot \nabla u_h + \int_{\Gamma_i} [\![\hat{u}(u_h) - u_h]\!] \cdot \{\vec{\tau}_h\} + \int_{\Gamma_i} \{\hat{u}(u_h) - u_h\}[\![\vec{\tau}_h]\!] + \int_{\partial\Omega} (u^b(u_h) - u_h) \vec{\tau}_h \cdot \vec{n}, \quad \forall \vec{\tau}_h \in [\mathcal{V}_h^p]^n. \tag{4.3}$$

Define the lifting operators $\vec{r}_h$ and $\vec{\ell}_h$ (see e.g. [3]) by the following problems: find $\vec{r}_h(u_h) \in [\mathcal{V}_h^p]^n$ and $\vec{\ell}_h(u_h) \in [\mathcal{V}_h^p]^n$ such that

$$\sum_{\kappa \in T_h} \int_\kappa \vec{\tau}_h \cdot \vec{r}_h(u_h) = - \int_{\Gamma_i} [\![\hat{u}(u_h) - u_h]\!] \cdot \{\vec{\tau}_h\} \tag{4.4}$$

$$- \int_{\partial\Omega} (u^b(u_h) - u_h)\vec{\tau}_h \cdot \vec{n}, \quad \forall \vec{\tau}_h \in [\mathcal{V}_h^p]^n ,$$

$$\sum_{\kappa \in T_h} \int_\kappa \vec{\tau}_h \cdot \vec{\ell}_h(u_h) = - \int_{\Gamma_i} \{\hat{u}(u_h) - u_h\} [\![\vec{\tau}_h]\!], \quad \forall \vec{\tau}_h \in [\mathcal{V}_h^p]^n . \tag{4.5}$$

Then, using (4.3) gives

$$\vec{g}_h = \nabla u_h - \vec{r}_h(u_h) - \vec{\ell}_h(u_h). \tag{4.6}$$

Substituting (4.6) into (4.1) gives the following discretization: find $u_h \in \mathcal{V}_h^p$ such that

$$B_h(u_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h f(u_h, \nabla u_h - \vec{r}_h(u_h) - \vec{\ell}_h(u_h)) = 0, \quad \forall v_h \in \mathcal{V}_h^p, \tag{4.7}$$

where $\vec{r}_h$ and $\vec{\ell}_h$ are defined in (4.4) and (4.5), respectively. To complete the scheme, one must define the numerical flux functions $\hat{u}$ and u^b. For the remainder of the paper, it is assumed that these fluxes have the following properties:

1. for each interior edge, e, there exists a constant vector $\vec{d}_e$ such that $\hat{u}(w_h) = \{w_h\} + \vec{d}_e \cdot [\![w_h]\!]$,
2. $u^b = 0$.

Remark 4.1. While the assumptions on the numerical fluxes may seem restrictive, to the best of the authors' knowledge, all existing mixed method formulations that are consistent use fluxes satisfying these assumptions for problems with homogeneous Dirichlet boundary conditions [3].

4.2. Dual Consistency Analysis. In this section, the dual consistency of the mixed formulation defined in Section 4.1 is considered. The dual consistency of this discretization is analyzed in two parts. First, a restrictive condition is assumed which implies that the scheme is dual consistent. Then, the proof is extended to the general case, where only asymptotic dual consistency can be shown.

PROPOSITION 4.1. *Let $D_{\nabla u} f(u, \nabla u)\psi \in [\mathcal{V}_h^p]^n$. Then, the DG formulation defined in* (4.7) *together with the discrete functional $\mathcal{J}_h$ defined in* (3.5) *forms a dual consistent discretization.*

Proof. Noting that the lifting operators $\vec{r}_h$ and $\vec{\ell}_h$ are linear functionals and that $\vec{r}_h(u) = \vec{\ell}_h(u) = 0$, linearizing R_h about the exact solution gives

$$R_h'[u](w_h, v_h) = B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa v_h D_1 f(u, \nabla u) w_h$$
$$- \sum_{\kappa \in T_h} \int_\kappa v_h D_{\nabla u} f(u, \nabla u) \cdot (\nabla w_h - \vec{r}_h(w_h) - \vec{\ell}_h(w_h)).$$

Thus, integrating by parts gives

$$\begin{aligned} R_h'[u](w_h, v_h) = {} & B_h(w_h, v_h) - \sum_{\kappa \in T_h} \int_\kappa w_h D_1 f(u, \nabla u) v_h \\ & + \sum_{\kappa \in T_h} \int_\kappa w_h \nabla \cdot (D_{\nabla u} f(u, \nabla u) v_h) - \int_{\partial\Omega} w_h (D_{\nabla u} f(u, \nabla u) v_h) \cdot \vec{n} \\ & - \int_{\Gamma_i} ([\![w_h]\!] \cdot \{D_{\nabla u} f(u, \nabla u) v_h\} + \{w_h\} [\![D_{\nabla u} f(u, \nabla u) v_h]\!]) \\ & + \sum_{\kappa \in T_h} \int_\kappa v_h D_{\nabla u} f(u, \nabla u) \cdot (\vec{r}_h(w_h) + \vec{\ell}_h(w_h)). \end{aligned} \tag{4.8}$$

Using the assumptions on $\hat{u}$ and u^b combined with the hypothesis that $D_{\nabla u} f(u, \nabla u)\psi \in [\mathcal{V}_h^p]^n$,

$$\sum_{\kappa \in T_h} \int_\kappa (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{r}_h(w_h) = \int_{\Gamma_i} [\![w_h]\!] \cdot \{D_{\nabla u} f(u, \nabla u)\psi\} + \int_{\partial\Omega} w_h (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{n}, \tag{4.9}$$

$$\sum_{\kappa \in T_h} \int_\kappa (D_{\nabla u} f(u, \nabla u)\psi) \cdot \vec{\ell}_h(w_h) = -\int_{\Gamma_i} (\vec{d}_e \cdot [\![w_h]\!]) [\![D_{\nabla u} f(u, \nabla u)\psi]\!]. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.8) gives

$$R'_h[u](w_h, \psi) = B_h(w_h, \psi) - \sum_{\kappa \in T_h} \int_\kappa w_h \left(D_1 f(u, \nabla u)\psi - \nabla \cdot (D_{\nabla u} f(u, \nabla u)\psi) \right) - \int_{\Gamma_i} \left(\hat{u}(w_h) [\![D_{\nabla u} f(u, \nabla u)\psi]\!] \right).$$

Furthermore, by the assumptions on the smoothness of u, ψ, and f, $[\![D_{\nabla u} f(u, \nabla u)\psi]\!] = 0$. Thus,

$$R'_h[u](v_h, \psi) = B_h(v_h, \psi) - \sum_{\kappa \in T_h} \int_\kappa v_h \left(D_1 f(u, \nabla u)\psi - \nabla \cdot (D_{\nabla u} f(u, \nabla u)\psi) \right).$$

Finally,

$$R'_h[u](v_h, \psi) - \mathcal{J}'_h[u](v_h) = 0, \quad \forall v_h \in \mathcal{W}_h^p,$$

which implies that the scheme is dual consistent. □

Of course, in general, the assumption of $D_{\nabla u} f(u, \nabla u)\psi \in [\mathcal{V}_h^p]^n$ is not realistic. For $D_{\nabla u} f(u, \nabla u)\psi \notin [\mathcal{V}_h^p]^n$, Proposition 4.1 does not hold. In this case, the mixed formulation is only asymptotically dual consistent.

PROPOSITION 4.2. *If $D_{\nabla u} f(u, \nabla u)\psi \in [H^{k+1}(\Omega)]^n$, where $1 \le k \le p$, then the DG discretization defined in* (4.7) *together with the functional $\mathcal{J}_h$ defined in* (3.5) *forms an asymptotically dual consistent discretization.*

To simplify the proof, some preliminary notation and lemmas are required. In particular, let $\mathcal{E}_h$ denote the set of all faces in the triangulation T_h. Define the jump operator, $[\![\cdot]\!]$, on boundary faces by $[\![s]\!] = s\vec{n}$ for scalar quantities and $[\![\vec{v}]\!] = \vec{v} \cdot \vec{n}$ for vector quantities. Define the average operator, $\{\cdot\}$, on boundary faces by $\{\vec{v}\} = \vec{v}$.

For the following lemmas, it is assumed that $\mathcal{W}_h^p = (\mathcal{V}_h^p + H^2(\Omega)) \cap H_0^1(\Omega)$. Furthermore, it is assumed that the set of triangulations, $[T_h]_{h>0}$, is quasi-uniform (see [15, 24] for definition).

LEMMA 4.3. *There exists a norm, $|\!|\!| \cdot |\!|\!|_* : \mathcal{W}_h^p \to \mathbb{R}$, and a constant, c, such that*

$$h^{-1/2} \sum_{e \in \mathcal{E}_h} \| [\![v]\!] \|_{0,e} \le \sum_{e \in \mathcal{E}_h} h_{\kappa_e}^{-1/2} \| [\![v]\!] \|_{0,e} \le c |\!|\!| v |\!|\!|_*, \quad \forall v \in \mathcal{W}_h^p,$$

where κ_e is such that $e \subset \partial\kappa_e$, $h = \max_{\kappa \in T_h} h_\kappa$, and $h_\kappa = \sup_{x,y \in \kappa} |x - y|$.

Proof. An example of such a norm is used by Arnold *et al.* [3]. In particular,

$$|\!|\!| v |\!|\!|_*^2 \equiv \sum_{\kappa \in T_h} (|v|_{1,\kappa}^2 + h_\kappa^2 |v|_{2,\kappa}^2) + \sum_{e \in \mathcal{E}_h} \| r_e([\![v]\!]) \|_{0,\Omega}^2,$$

where

$$\int_\Omega r_e(\llbracket v \rrbracket) \cdot \vec{\tau} = -\int_e \llbracket v \rrbracket \cdot \{\vec{\tau}\} \quad \forall \vec{\tau} \in [\mathcal{V}_h^p]^n.$$

For the proof of the lemma for this norm, see [3], Section 4.1, or [10], Lemma 2. Note that only the existence of such a norm, not its particular form, is important here. □

LEMMA 4.4. *For a face, $e \in \mathcal{E}_h$, such that $e \subset \partial\kappa$, there exists a constant, c, such that, for all $v \in H^1(\kappa)$ and $w \in L^2(e)$,*

$$\int_e |vw| \le c h_\kappa^{-1/2} (\|v\|_{0,\kappa} + h_\kappa |v|_{1,\kappa}) \|w\|_{0,e}.$$

Proof. Apply the Cauchy-Schwarz inequality, and then use

$$\|v\|_{0,e} \le c h_\kappa^{-1/2} (\|v\|_{0,\kappa} + h_\kappa |v|_{1,\kappa}), \quad \forall v \in H^1(\kappa),$$

which is a standard trace theorem [2]. □

LEMMA 4.5. *For all $w \in H^{k+1}(\Omega)$, where $1 \le k \le p$, there exists a constant, c, such that*

$$\left(\sum_{\kappa \in T_h} \|w - \Pi_h^p(w)\|_{1,\kappa}^2 \right)^{1/2} \le c h^k |w|_{k+1,\Omega},$$

where $\Pi_h^p : L^2(\Omega) \to \mathcal{V}_h^p$ is the $L^2(\Omega)$-orthogonal projection onto $\mathcal{V}_h^p$.

Proof. If $\Pi_\kappa^p : L^2(\kappa) \to P^p$ is the $L^2(\kappa)$-orthogonal projection onto P^p, then

$$\Pi_h^p(v)|_\kappa = \Pi_\kappa^p(v|_\kappa), \quad \forall v \in L^2(\Omega).$$

To complete the proof, apply Proposition 1.134(iii) from [15] to each element κ and sum over the elements. □

Proof. [Proposition 4.2] Define $\vec{\pi} \in [\mathcal{V}_h^p]^n$ by

$$\pi_j = \Pi_h^p((D_{\nabla u} f(u, \nabla u)\psi)_j), \quad \text{for } j = 1, \dots, n.$$

Furthermore, define $\vec{\epsilon} \in [L^2(\Omega)]^n$ by

$$\epsilon_j = (D_{\nabla u} f(u, \nabla u)\psi)_j - \pi_j, \quad \text{for } j = 1, \dots, n.$$

By assumption, $D_{\nabla u} f(u, \nabla u)\psi \in [H^1(\Omega)]^n$. Thus, $\vec{\epsilon}|_\kappa \in [H^1(\kappa)]^n$, for all $\kappa \in T_h$. From the proof of Proposition 4.1, it is clear that

$$E_h(v_h) \equiv R_h'[u](v_h, \psi) - \mathcal{J}_h'[u](v_h) = \sum_{\kappa \in T_h} \int_\kappa (D_{\nabla u} f(u, \nabla u)\psi) \cdot \left(\vec{r}_h(v_h) + \vec{\ell}_h(v_h) \right) - \int_\Gamma \llbracket v_h \rrbracket \cdot (D_{\nabla u} f(u, \nabla u)\psi), \quad \forall v_h \in \mathcal{W}_h^p,$$

where $\Gamma \equiv \Gamma_i \cup \partial\Omega$. Thus,

$$E_h(v_h) = \sum_{\kappa \in T_h} \int_\kappa (\vec{\pi} + \vec{\epsilon}) \cdot \left(\vec{r}_h(v_h) + \vec{\ell}_h(v_h) \right) - \int_\Gamma \llbracket v_h \rrbracket \cdot (\vec{\pi} + \vec{\epsilon}), \quad \forall v_h \in \mathcal{W}_h^p. \tag{4.11}$$

From (4.4), (4.5), and the assumptions on $\hat{u}$ and u^b,

$$\sum_{\kappa\in T_h}\int_\kappa \vec{\pi}\cdot\vec{r}_h(v_h) = \int_\Gamma [\![v_h]\!]\cdot\{\vec{\pi}\}, \tag{4.12}$$

$$\sum_{\kappa\in T_h}\int_\kappa \vec{\pi}\cdot\vec{\ell}_h(v_h) = -\int_{\Gamma_i} (\vec{d}_e\cdot[\![v_h]\!])[\![\vec{\pi}]\!]. \tag{4.13}$$

Substituting (4.12) and (4.13) into (4.11) gives

$$E_h(v_h) = \sum_{\kappa\in T_h}\int_\kappa \vec{\epsilon}\cdot(\vec{r}_h(v_h)+\vec{\ell}_h(v_h)) - \int_\Gamma [\![v_h]\!]\cdot\{\vec{\epsilon}\} - \int_{\Gamma_i}(\vec{d}_e\cdot[\![v_h]\!])[\![\vec{\pi}]\!].$$

By the definition of $\vec{\epsilon}$,

$$\sum_{\kappa\in T_h}\int_\kappa \vec{\epsilon}\cdot\vec{z}_h = 0, \quad \forall \vec{z}_h\in[\mathcal{V}_h^p]^n.$$

Furthermore, since $[\![D_{\nabla u}f(u,\nabla u)\psi]\!]=0$, it is clear that $[\![\vec{\pi}]\!] = -[\![\vec{\epsilon}]\!]$. Thus,

$$E_h(v_h) = -\int_\Gamma [\![v_h]\!]\cdot\{\vec{\epsilon}\} + \int_{\Gamma_i}(\vec{d}_e\cdot[\![v_h]\!])[\![\vec{\epsilon}]\!].$$

Then, using the assumption on $\hat{u}$, $E_h(v_h)$ can be rewritten as

$$E_h(v_h) = -\int_{\partial\Omega} v_h\vec{\epsilon}\cdot\vec{n} - \int_{\Gamma_i}[\![v_h]\!]\cdot\left[\left(\frac{1}{2}-\vec{d}_e\cdot\vec{n}^+\right)\vec{\epsilon}^+ + \left(\frac{1}{2}+\vec{d}_e\cdot\vec{n}^+\right)\vec{\epsilon}^-\right],$$

where $(\cdot)^+$ and $(\cdot)^-$ refer to trace values taken from opposite sides of an interior face, and $\vec{n}^+$ is the outward pointing unit normal for κ^+.

Applying Lemma 4.4 to each edge in the triangulation, one can show that

$$|E_h(v_h)| \le \sum_{e\in\mathcal{E}_h}\sum_{j=1}^n\left[c_e h_{\kappa_e}^{-1/2}\left(\|\epsilon_j\|_{0,\kappa_e} + h_{\kappa_e}|\epsilon_j|_{1,\kappa_e}\right)\|[\![v_h]\!]\|_{0,e}\right].$$

Thus,

$$\begin{aligned}
|E_h(v_h)| &\le \sum_{e\in\mathcal{E}_h}\sum_{j=1}^n\left[c_e h_{\kappa_e}^{-1/2}\left(\|\epsilon_j\|_{1,\kappa_e}^2\right)^{1/2}\|[\![v_h]\!]\|_{0,e}\right],\\
&\le \sum_{e\in\mathcal{E}_h}\sum_{j=1}^n\left[c_e h_{\kappa_e}^{-1/2}\left(\sum_{\kappa\in T_h}\|\epsilon_j\|_{1,\kappa}^2\right)^{1/2}\|[\![v_h]\!]\|_{0,e}\right],\\
&\le C\sum_{j=1}^n\left[\left(\sum_{\kappa\in T_h}\|\epsilon_j\|_{1,\kappa}^2\right)^{1/2}\right]\times\left[\sum_{e\in\mathcal{E}_h}h_{\kappa_e}^{-1/2}\|[\![v_h]\!]\|_{0,e}\right].
\end{aligned}$$

Finally, applying Lemmas 4.3 and 4.5 gives

$$|E_h(v_h)| \le C|\!|\!|v_h|\!|\!|_* h^k\sum_{j=1}^n|(D_{\nabla u}f(u,\nabla u)\psi)_j|_{k+1,\Omega}.$$

Thus, as $h\to 0$, $|E_h(v_h)|\to 0$ for all $v_h\in\mathcal{W}_h^p$, which implies that the scheme is asymptotically dual consistent. □

5. Numerical Results. As a demonstration of the effects of dual consistency, a simple test problem based on a nonlinear ODE is considered. The effect of dual consistency on the convergence rates of the solution and adjoint solution errors as well as a simple functional output is demonstrated.

Consider the following ODE:

$$-((\nu+u)u_x)_x - cu_xu_x = g \quad \text{for } x \in (0,1),$$
$$u(0) = u(1) = 0,$$

where $\nu = 1$ and $c = \frac{1}{2}$. Setting

$$g(x) = \pi^2((\nu + \sin(\pi x))\sin(\pi x) - (1+c)\cos^2(\pi x)),$$

it is easy to show that the exact solution is given by

$$u_e(x) = \sin(\pi x).$$

This nonlinear problem has been solved using three discretizations: the standard weighting method as shown in (3.4), a dual consistent method with a penalty term as shown in (3.8), and an asymptotically dual consistent mixed method with $\hat{u} = \{u\}$. In all cases, the BR2 scheme is used to discretize the nonlinear diffusion operator.

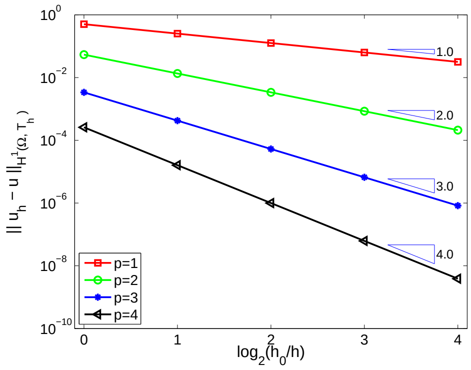
Figure 5.1 shows the error in the primal solution versus grid refinement. The error is measured in a broken H^1 norm defined by

$$\|v\|^2_{H^1(\Omega,T_h)} = \sum_{\kappa\in T_h} \int_\kappa (v^2 + v_x^2).$$

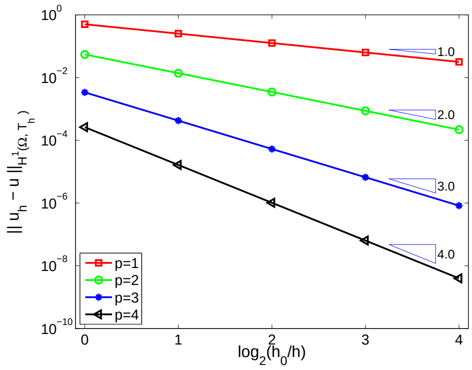
In this norm, all three schemes produce essentially the same error in the primal solution. However, as shown in Figure 5.2, the dual consistent and asymptotically dual consistent schemes produce superior results when the error is measured in the L^2 norm. In particular, for the dual inconsistent discretization, the L^2 norm of the error is proportional to h^p for even p and proportional to h^{p+1} for odd p. It is not clear why the even and odd p results show different asymptotic rates, but similar results have been observed for other dual inconsistent discretizations [19].

Alternatively, the dual consistent and asymptotically dual consistent discretizations give $O(h^{p+1})$ for all p tested. Furthermore, it is interesting to note that the asymptotically dual consistent method produces essentially exactly the same results as the dual consistent discretization. This fact shows that, for the asymptotically dual consistent scheme, the contribution of the dual inconsistency error to the total error is sufficiently high-order that it does not degrade the asymptotic rate convergence rate of the L^2 error. This result is not surprising given the form of the dual consistency error derived in Section 4.

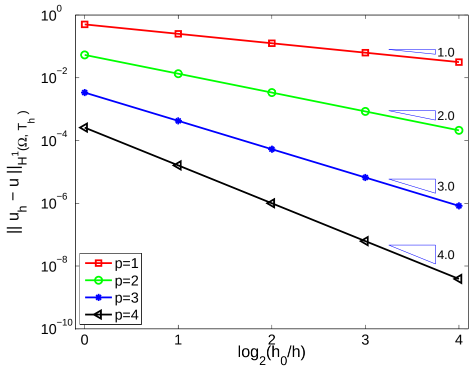
Examining the adjoint solution error, one can see that the dual consistent and asymptotically dual consistent schemes are superior for computing the adjoint. Figure 5.3 shows the adjoint error in the broken H^1 norm. The adjoint error is computed relative to a 40th order solution of a Galerkin spectral discretization of the dual problem. When using the dual inconsistent discretization, the broken H^1 norm of the adjoint error does not converge to zero with grid refinement. For the dual consistent and asymptotically dual consistent schemes, this error converges at $O(h^p)$. Similarly, Figure 5.4 shows that the L^2 norm of the adjoint error converges at $O(h)$ when using the dual inconsistent scheme, regardless of p, while, for the dual consistent and asymptotically dual consistent schemes, this error converges at $O(h^{p+1})$.



(a) Dual consistent



(b) Dual inconsistent



(c) Asymptotically dual consistent

FIG. 5.1. *Primal error in the broken H^1 norm*

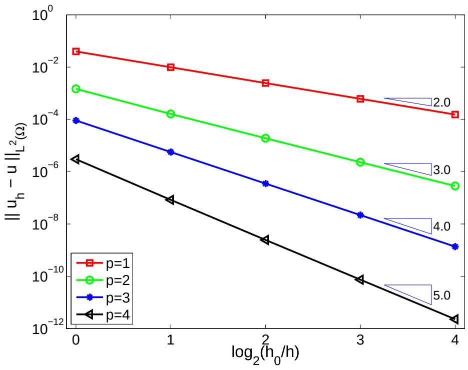
To further understand the differences in the adjoint solutions for the different discretizations, Figure 5.5 shows the adjoint solution for each scheme, computed with $p = 2$ polynomials on an eight element mesh. At the scale shown, it is difficult to see the difference between the discretizations. However, examining the pointwise error relative to the 40th order spectral solution, shown in Figure 5.6, it is clear that the adjoint solution for the dual inconsistent scheme is more oscillatory than that for the dual consistent or asymptotically dual consistent discretizations.

Finally, let

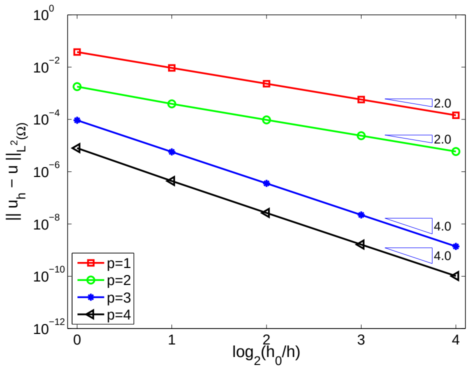
$$\mathcal{J}(u) = \frac{1}{2}\int_0^1 (w - u)^2,$$

where $w(x) = 2\sin(\pi x)$, be the output of interest. Then, computing the exact functional output is trivial, enabling comparison of the computed result with the exact value, $\mathcal{J}(u_e) = 1/4$.

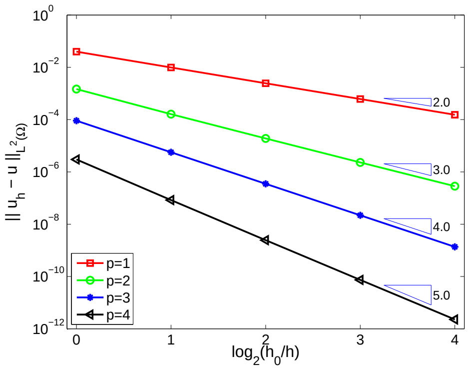
Figure 5.7 shows the error in the computed functional for the three discretizations considered. The figure shows that, as in the state and adjoint results, the performance



(a) Dual consistent



(b) Dual inconsistent

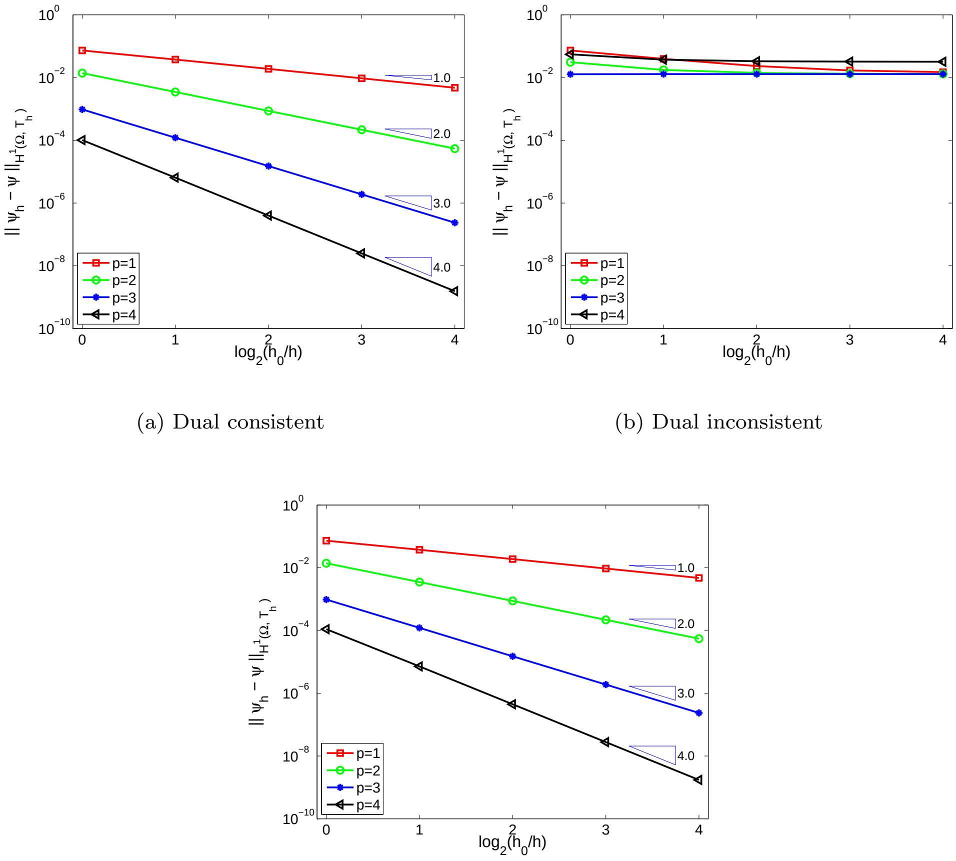


(c) Asymptotically dual consistent

FIG. 5.2. *Primal error in the L^2 norm*

the dual consistent and asymptotically dual consistent schemes is very similar. Both schemes achieve $O(h^{2p})$ for $1 \leq p \leq 4$. However, for the dual inconsistent scheme, the convergence rate of the functional is $O(h^p)$ for even p and $O(h^{p+1})$ for odd p. Thus, the dual consistent and asymptotically dual consistent discretizations predict the functional with greater accuracy than the dual inconsistent discretization for similar numbers of degrees of freedom.

6. Conclusions. The effect of dual consistency on DG discretizations of solution and solution gradient dependent source terms has been examined. In particular, the standard weighting DG discretization of source terms depending on the gradient of the solution has been analyzed and shown to be dual inconsistent. Starting from this dual inconsistent scheme a dual consistent discretization has been developed. Furthermore, discretizations derived using the mixed formulation have been shown to be asymptotically dual consistent. Numerical results from a simple test problem demonstrate that the dual consistent and asymptotically dual consistent schemes are superior both in terms of solution accuracy and output accuracy.



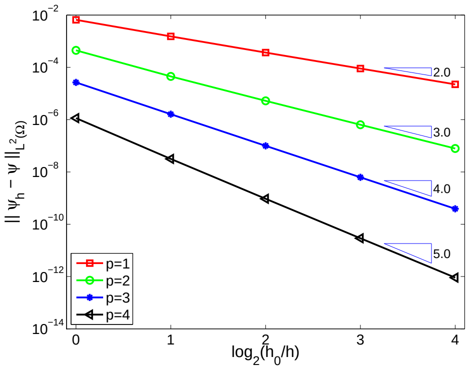
(a) Dual consistent

(b) Dual inconsistent

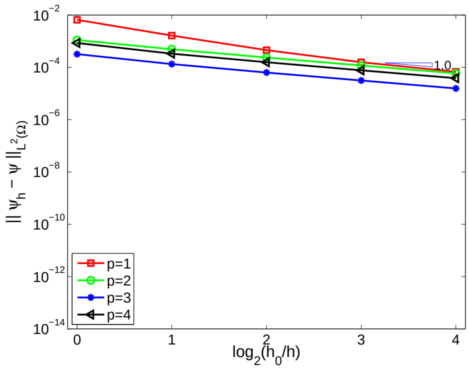
(c) Asymptotically dual consistent

FIG. 5.3. *Adjoint error in the broken* H^1 *norm*

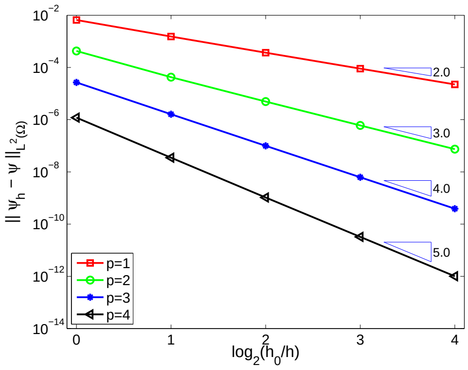
Further work is required in many areas. In particular, this work has considered only the effect of dual consistency. Using the methods presented in Section 3 and Section 4, one could construct a consistent and dual consistent but unstable scheme. Thus, while techniques for constructing a consistent and dual consistent (or asymptotically dual consistent) discretization have been shown, a method for ensuring that the resulting scheme is stable is left for future research. Finally, given the extremely similar results shown for the dual consistent and asymptotically dual consistent schemes considered, it remains to be determined which of these schemes is most effective for practical problems.



(a) Dual consistent



(b) Dual inconsistent



(c) Asymptotically dual consistent

FIG. 5.4. *Adjoint error in the L^2 norm*

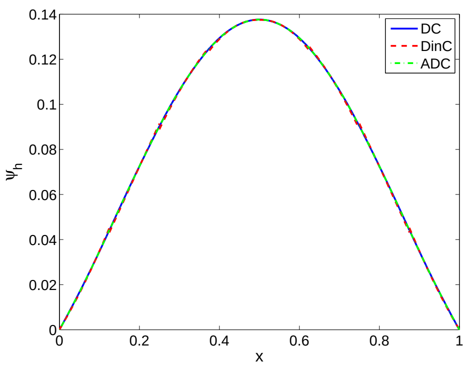
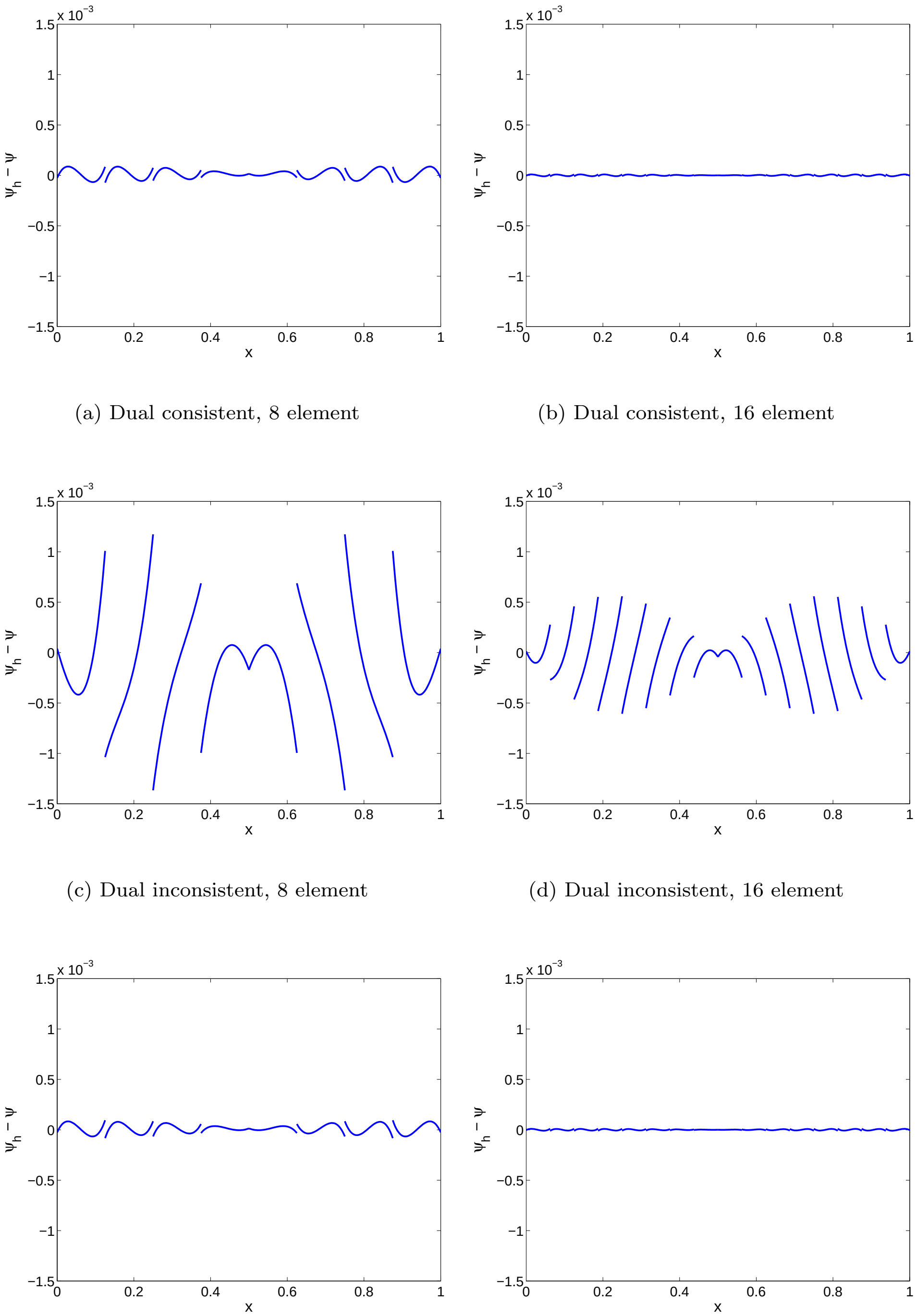


FIG. 5.5. *Adjoint solution computed using the $p = 2$ dual consistent (DC), dual inconsistent (DinC), and asymptotically dual consistent (ADC) discretizations on an eight element mesh*



(a) Dual consistent, 8 element

(b) Dual consistent, 16 element

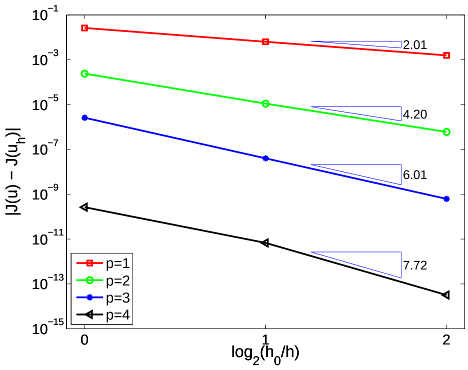
(c) Dual inconsistent, 8 element

(d) Dual inconsistent, 16 element

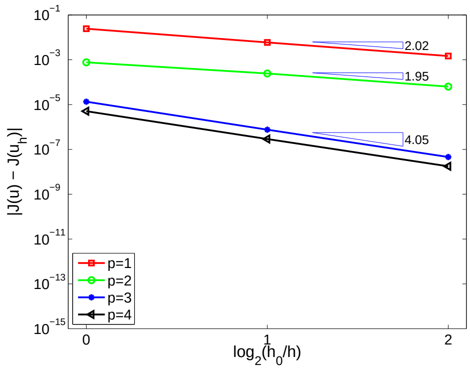
(e) Asym. dual consistent, 8 element

(f) Asym. dual consistent, 16 element

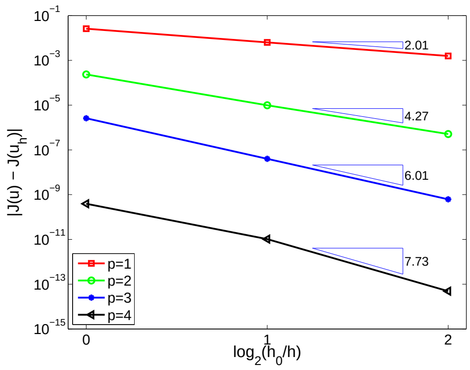
FIG. 5.6. *Adjoint error for $p = 2$ on eight and sixteen element meshes*



(a) Dual consistent



(b) Dual inconsistent



(c) Asymptotically dual consistent

FIG. 5.7. *Functional output error*

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