Asymptotic *a Posteriori* Finite Element Bounds for the Outputs of Noncoercive Problems: the Helmholtz and Burgers Equations*

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January 11, 1998

Abstract

We describe an *a posteriori* finite element procedure for the efficient computation of lower and upper estimators for linear–functional outputs of noncoercive linear and semilinear elliptic second–order partial differential equations. Under a relatively weak hypothesis related to the relative magnitude of the L^2 and H^1 errors of the reconstructed solution, these lower and upper estimators converge to the true output from below and above, respectively, and thus constitute asymptotic bounds. In numerical experiments we find that our hypothesis is satisfied once the finite element triangulation even roughly resolves the structure of the exact solution, and thus, in practice, the bounds prove quite reliable. Numerical results are presented for the one–dimensional Helmholtz equation and for the Burgers equation.

1 Introduction

In a recent series of papers [10, 11, 12, 13] we introduce an *a posteriori* finite element method for the efficient computation of strict upper and lower bounds for linear–functional outputs of coercive partial differential equations. Our technique, a generalization of earlier *a posteriori* bounds for

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the error in the energy norm [8, 4, 2], constitutes an implicit alternative to Aubin–Nitsche explicit linear–functional *a posteriori* estimators [5, 6, 14]. The advantage of our approach is that we obtain rigorous, accurate, and quantitative (constant–free) bounds directly for the engineering quantities of interest; the former provides certainty, while the latter ensures relevance. The methods are thus appropriate for both confirmation [10, 11, 12] and adaptivity [13].

In our earlier papers, we apply our bound techniques primarily to (symmetric and nonsymmetric) coercive linear operators. Although certain special noncoercive systems are treated — for example, the Stokes equations with discontinuous pressure approximations [12, 9] — and certain nonlinear problems are also addressed — for example, the Poisson operator with a cubic nonlinearity [12] — more general noncoercive equations are not considered. In this short paper we show that, under a very plausible hypothesis, our procedure can be directly applied to noncoercive problems — and rigorous upper and lower bounds will still obtain. Disappointingly, we can not verify the hypothesis directly, and there is, therefore, an undeniable loss of certainty with respect to the coercive framework. However, the hypothesis will typically be satisfied for any finite element triangulation that even roughly resolves the structure of the exact solution, and thus, in practice, the bounds should prove quite reliable. It is not clear that, for general noncoercive problems, any greater certainty is possible.

Our hypothesis — a variation on the standard Aubin–Nitsche relationship between the L^2 and H^1 discretization errors — applies quite generally to second–order equations that satisfy a Gårding inequality [15]. It is well established that many *a posteriori* procedures are relatively insensitive to lower–derivative linear or nonlinear terms [4, 16]; in particular, even if these terms are neglected in the construction of the *a posteriori* operator that inverts the residual, asymptotic exactness will still typically obtain. Our results may thus be interpreted as a rather natural generalization of this "L^2–insensitivity" to the case of linear–functional outputs. It is, however, somewhat surprising that our procedure yields estimators that are not only asymptotically exact, but also asymptotically bounds: our lower estimator must approach the true output from below, and our upper estimator must approach the true output from above. It directly follows that the estimator gap, the difference between the upper and lower estimators, asymptotically approaches the true error from above, thus providing the requisite conservative bound.

In this paper we present the formulation and proof of asymptotic bounds for two one–dimensional problems, the Helmholtz equation (in Section 2) and the Burgers equation (in Section 3); we also present numerical results on uniform meshes to demonstrate the relatively weak nature of our hypothesis. Our focus here is on the essential idea that allows us to extend our earlier work to the noncoercive case. Future papers will consider the extension to higher dimensions, and to the development of local indicators and associated adaptive strategies; in both instances our techniques for coercive equations [11, 13] directly apply, although there are certainly new phenomena that arise in the dispersive context [7, 3]. We note that our hypothesis will be even less problematic in the adaptive context — refinement will naturally drive the finite element solution towards the asymptotic state.

2 The Helmholtz Equation

2.1 Problem Formulation

We consider the real–valued one–dimensional Helmholtz equation (reduced wave equation) on $\Omega =]0, 1[$,

$$- u_{xx} - k^2 u = f \text{ in } \Omega, \tag{1}$$

with Dirichlet boundary conditions, $u(0) = b_0$, $u(1) = b_1$. We assume that the real positive constant k^2 is chosen such that the problem (1) is well–posed.

We next introduce the weak form of this problem, which is the point of departure for the finite element discretization. Let

$$X_D = \{v \in H^1(\Omega) \,|\, v(0) = b_0, v(1) = b_1\}, \tag{2}$$

$$X = \{v \in H^1(\Omega) \,|\, v(0) = v(1) = 0\}, \tag{3}$$

where $H^1(\Omega)$ is the usual Sobolev space [1]. We then look for $u \in X_D$ such that

$$a(v, u) = \langle f, v\rangle, \ \forall v \in X, \tag{4}$$

where

$$a(v, w) = A(v, w) - B(v, w), \tag{5}$$

and

$$A(v, w) = \int_0^1 v_x w_x \, dx, \tag{6}$$

$$B(v,w) = k^2 \int_0^1 vw \, dx \ . \tag{7}$$

Here $\langle \, , \rangle$ is the duality pairing between $H^1(\Omega)$ and $H^{-1}(\Omega)$. Consistent with our hypothesis of well–posedness, we assume that a is characterized by a nonzero inf–sup parameter [7].

Finally, the output of interest, s, is given by

$$s = \ell^O(u), \tag{8}$$

where $\ell^O(v)$ is a bounded linear functional. For example, $\ell(v) = v(1/2)$ yields $s = u(1/2)$ as the quantity of interest.

2.2 Finite Element Spaces and Forms

We introduce two "triangulations" of Ω: a working mesh $\mathcal{T}_H$, and a truth mesh $\mathcal{T}_h$. We assume that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$, in that any element T_h of $\mathcal{T}_h$ lies entirely within a single element T_H of $\mathcal{T}_H$. In this paper we consider only uniform meshes: $\mathcal{T}_H$ consists of N_H elements of length $H = 1/N_H$; $\mathcal{T}_h$ consists of N_h elements of length $h = 1/N_h$.

We can then define the standard piecewise–linear finite element spaces in the usual fashion,

$$X_\delta = \{v|_{T_\delta} \in I\!P_1(T_\delta), \forall T_\delta \in \mathcal{T}_\delta\} \cap X, \tag{9}$$

$$X_{D\,\delta} = \{v|_{T_\delta} \in I\!P_1(T_\delta), \forall T_\delta \in \mathcal{T}_\delta\} \cap X_D, \tag{10}$$

where $I\!P_1(T_\delta)$ refers to the space of linear polynomials over T_δ. The working discretization corresponds to $\delta = H$; the truth discretization corresponds to $\delta = h$. Note that, by virtue of the refinement hypothesis, $X_{D\,H} \subset X_{D\,h}$, and $X_H \subset X_h$.

We shall need several additional, less standard, spaces for the bound procedure [10, 11]. First, we require the "broken" versions of X_δ with respect to the domain decomposition induced by X_H,

$$\hat{X}_\delta = \{v|_{T_\delta} \in I\!P_1(T_\delta), \forall T_\delta \in \mathcal{T}_\delta \,|\, v|_{T_H} \in H^1(T_H), \forall T_H \in \mathcal{T}_H\}. \tag{11}$$

Second, we require a space defined on the nodes x_n, $n = 1, \ldots, N_H + 1$, of the coarse triangulation $\mathcal{T}_H$,

$$Q = I\!R^{N_H+1}, \tag{12}$$

where $I\!R$ denotes the reals. Note that $x_1 = 0$, and $x_{N_H+1} = 1$.

Finally, in addition to the bilinear and linear forms introduced in Section 2.1 — appropriately extended to the broken spaces $\hat{X}_H$ and $\hat{X}_h$ — we shall require a "continuity" bilinear form over $\hat{X}_h$ (or $\hat{X}_H$) $\times Q$,

$$b(v, q) = \sum_{n=1}^{N_H+1} q_n [v]_n . \tag{13}$$

Here $[v]_1 = v(0)$, $[v]_{N_H+1} = v(1)$, and $[v]_n$, $n = 2, \ldots, N_H$, is the difference between v approaching x_n from the right and v approaching x_n from the left. Note that $b(v, q)$ vanishes for any v in either X_H or X_h.

2.3 Finite Element Approximation

For the working discretization, we look for $u_H \in X_{D\,H}$ such that

$$a(v, u_H) = \langle f, v \rangle, \forall v \in X_H, \tag{14}$$

and then evaluate $s_H = \ell^O(u_H)$. For the truth discretization, we look for $u_h \in X_{D\,h}$ such that

$$a(v, u_h) = \langle f, v \rangle, \forall v \in X_h, \tag{15}$$

and then evaluate $s_h = \ell^O(u_h)$. We assume that both these discrete problems admit a unique solution. We further assume that h is sufficiently small that s_h differs negligibly from $s = \ell^O(u)$.

Our goal is to find, based solely on computations on the working discretization X_H and the *decoupled* truth discretization $\hat{X}_h$, lower and upper estimators that provide lower and upper bounds for s_h for all $H \leq H_0$ for some $H_0 > h$. The computational requirement ensures that (in higher space dimensions) the calculations are considerably less expensive than direct computation of s_h on the coupled truth discretization X_h; the bound requirement — in conjunction with our assumption that s_h is arbitrarily close to s — ensures that the results obtained are meaningful measures of the true output s_h. In practice, the estimators are of interest both as predictors of s_h, and — through the estimator gap — as indicators of the error in s_H; in local form, the latter serves to define adaptive strategies [13].

2.4 Bound Computation

We first find u_H from (14), and an output–adjoint $\psi_H \in X_H$ from

$$a(\psi_H, v) = -\ell^O(v), \forall v \in X_H. \tag{16}$$

We then compute the hybrid flux $p_H \in Q$ from

$$b(v, p_H) = \kappa R_H(v) - \ell^O(v) - a(\psi_H, v), \forall v \in \hat{X}_H, \tag{17}$$

where

$$R_H(v) \equiv \langle f, v \rangle - a(v, u_H) \tag{18}$$

is the residual, and κ is a non–negative real constant. For future reference we note that, from (14), $R_H(v) = 0, \forall v \in X_H$. It can be readily shown that (17) has a unique solution.

Next, we find $\hat{e}_h \in \hat{X}_h$ from

$$2\kappa a(v, \hat{e}_h) = \kappa R_H(v) - \ell^O(v) - a(\psi_H, v) - b(v, p_H), \forall v \in \hat{X}_h, \tag{19}$$

which we assume yields a unique solution; the latter is certainly plausible for kH of order unity or smaller. Then, based on $\hat{e}_h$, we compute our lower estimator

$$s_{LO} = \ell^O(u_H) - \kappa a(\hat{e}_h, \hat{e}_h). \tag{20}$$

As described in [10, 13], this lower estimator can be readily optimized (maximized) with respect to the scalar scaling parameter, κ; this capability will be exercised in the numerical results of Section 2.6. Finally, we may obtain a corresponding upper estimator for s_h simply by pursuing a lower estimator for $-\ell^O(u_h)$ [10, 11, 13].

2.5 Asymptotic Properties

We now define $e_h = u_h - u_H$, and $z_h = e_h - \hat{e}_h$. We then make the hypothesis that there exists an $H_0 > h$ such that, for all $H \leq H_0$,

$$B(z_h, z_h) \leq A(z_h, z_h), \tag{21}$$

or equivalently,

$$a(z_h, z_h) \geq 0. \tag{22}$$

Note that (21) is effectively an assumption on the L^2 error ($O(H^4)$) relative to the H^1 error ($O(H^2)$) of the reconstructed truth solution $u_H + \hat{e}_h$: in the presence of sufficient regularity, this result can be rigorously proven by application of the usual Aubin–Nitsche arguments — the essential point is that $a(v, z_h) = 0$ for all v in X_H. In practice, we expect — if interpolation theory is an adequate guide — that (21) will be satisfied as soon as the working mesh $\mathcal{T}_H$ even roughly resolves the structure of u, that is, $kH \approx 1$.

If the mesh is too coarse we expect that (21) will not be satisfied, except in the relatively uninteresting case in which k^2 is not large. Hypotheses similar to (21) are common in the analysis of noncoercive operators [15].

We now claim that, if (21) is satisfied, then $s_{LO} \leq \ell^O(u_h)$, as desired; a similar proof holds for the upper bound. The simplest proof (and, in fact, the most transparent derivation of the bound procedure) is based on duality arguments that differ from the demonstrations of [10, 11, 12, 13] only in the restriction of admissible candidates to the subset of $\hat{X}_h$ that satisfies our hypothesis (21). In the interest of brevity, we give here another proof that does not require the introduction of the duality framework and associated Lagrangian. This alternative procedure also yields a different interpretation for the role of, and requirements on, each of the terms in (19).

We first note from (19) that

$$2\kappa a(e_h, \hat{e}_h) = \kappa R_H(e_h) - \ell^O(e_h), \tag{23}$$

where we have evoked (i) $e_h \in X_h$ and the definition of b, (13), and (ii) the classical orthogonality relation, $a(v, e_h) = 0, \forall v \in X_H$, for $v = \psi_H$. Note in the latter we must appeal to our refinement assumption, $X_H \subset X_h$. We next note that, since e_h is in X_h, $R_H(e_h) \equiv \langle f, e_h \rangle - a(e_h, u_H) = a(e_h, e_h)$, which thus yields

$$2\kappa a(e_h, \hat{e}_h) = \kappa a(e_h, e_h) - \ell^O(e_h). \tag{24}$$

Finally, we observe that

$$\kappa a(z_h, z_h) = \kappa a(e_h, e_h) + \kappa a(\hat{e}_h, \hat{e}_h) - 2\kappa a(e_h, \hat{e}_h), \tag{25}$$

and thus from (24),

$$-\kappa a(\hat{e}_h, \hat{e}_h) = -\kappa a(z_h, z_h) + \ell^O(e_h). \tag{26}$$

It then directly follows from (20) that

$$\begin{aligned} s_{LO} &= \ell^O(u_h) - \ell^O(e_h) - \kappa a(\hat{e}_h, \hat{e}_h) \\ &= \ell^O(u_h) - \kappa a(z_h, z_h), \end{aligned} \tag{27}$$

and thus $s_{LO} \leq \ell(u_h)$ if (and only if) (21) is satisfied.

2.6 Numerical Results

We consider (1) with $k^2 = 400$, $b_0 = -1$, $b_1 = 0$, and $f = 0$; the output chosen is the value of u at $x = 0.5$. The truth mesh is given by $h = 1/10,240$; the working mesh diameter varies from $H = h$ to $H = 1/10$. Note that, on the coarsest working mesh, $kH = 2$, which clearly does not resolve the anticipated wave structure. For each working mesh H we compute the lower (s_{LO}) and upper (s_{UP}) estimators — optimized with respect to κ as described in [13] — the results of which are presented in Figure 1 in the form of s_{LO}/s_h and s_{UP}/s_h.

First, we observe that we obtain bounds even for the largest H. Independent computation of $A(z_h, z_h)$ and $B(z_h, z_h)$ confirms that (21) is satisfied; as expected, $A(z_h, z_h)$ vanishes as H^2, while $B(z_h, z_h)$ vanishes as H^4. Second, we observe that the bounds converge quadratically; in fact, this follows directly from the observed behavior of $A(z_h, z_h)$ and $B(z_h, z_h)$ and our result (27). Note that we obtain quadratic convergence for any κ, however the bounds are much sharper for the optimal value of κ; for this particular output, the error in the bounds — the bound gap — for $\kappa = 1$ is almost a factor of ten larger than the bound gap obtained for the optimal value of κ. Third, we emphasize that there are no undetermined constants in the results presented in Figure 1; furthermore, given the apparently robust hypothesis (21), and the assumption that s_h is sufficiently close to s, the bounds are rigorous. Fourth, and finally, we reiterate that the bounds are for the actual quantity of interest, and are thus directly relevant to engineering design and optimization.

3 The Burgers Equation

3.1 Problem Formulation

We consider the one–dimensional Burgers equation on $\Omega =]0, 1[$,

$$- \nu u_{xx} + \frac{1}{2}(u^2)_x = f \text{ in } \Omega, \tag{28}$$

with Dirichlet boundary conditions, $u(0) = b_0$, $u(1) = b_1$. Here ν, the viscosity, is a real positive constant.

We next introduce the weak form of this problem, which is the point of departure for the finite element discretization. The spaces X_D and X are defined precisely as for the Helmholtz problem. We then look for $u \in X_D$

such that

$$\nu(v_x, u_x; 1) - \frac{1}{2}(v_x, u^2; 1) = \langle f, v \rangle, \ \forall v \in X, \tag{29}$$

where $(\, , \, ; \omega)$ denotes the ω–"weighted" L^2 inner product over Ω,

$$(v, w; \omega) = \int_0^1 vw \, \omega \, dx, \tag{30}$$

and $\langle \, , \, \rangle$ is, as before, the duality pairing between $H^1(\Omega)$ and $H^{-1}(\Omega)$. The output of interest, s, is given by

$$s = \ell^O(u), \tag{31}$$

where $\ell^O(v)$ is a bounded linear functional.

3.2 Finite Element Spaces and Forms

Our triangulations $\mathcal{T}_H$ and $\mathcal{T}_h$ are defined exactly as for the Helmholtz problem, as are the associated finite element spaces, $X_{D\,H}$ and X_H, and $X_{D\,h}$ and X_h, respectively, the corresponding broken spaces, $\hat{X}_H$ and $\hat{X}_h$, respectively, and the hybrid flux space, Q. Similarly, the continuity bilinear form b, (13), requires no modification.

For the nonlinear Burgers operator it shall be simplest to define most bilinear forms directly in terms of the weighted L^2 inner product (30). We shall, however, define a_H, A, and B_H as

$$a_H(v, w) = A(v, w) - B_H(v, w), \tag{32}$$

where

$$A(v, w) = \nu(v_x, w_x; 1) \tag{33}$$

and

$$B_H(v, w) = \frac{1}{2}(v, w; \psi_{H\,x} - u_{H\,x}), \tag{34}$$

where u_H and ψ_H are the working–discretization solution and output–adjoint, respectively. Note that A and B_H for the Burgers equation are similar in nature to the corresponding bilinear forms for the Helmholtz equation: A measures the H^1 norm, and B_H measures a weighted L^2 norm. In the case of the nonlinear Burgers equation, however, a_H depends on the approximate solution.

3.3 Finite Element Approximation

For the working discretization, we look for $u_H \in X_{D\,H}$ such that

$$\nu(v_x, u_{H\,x}; 1) - \frac{1}{2}(v_x, u_H^2; 1) = \langle f, v\rangle, \forall v \in X_H, \tag{35}$$

and then evaluate $s_H = \ell^O(u_H)$. For the truth discretization, we look for $u_h \in X_{D\,h}$ such that

$$\nu(v_x, u_{h\,x}; 1) - \frac{1}{2}(v_x, u_h^2; 1) = \langle f, v\rangle, \forall v \in X_h, \tag{36}$$

and then evaluate $s_h = \ell^O(u_h)$. We assume that both these discrete problems admit a unique solution. We further assume, as for the Helmholtz problem, that h is sufficiently small that s_h differs negligibly from $s = \ell^O(u)$.

3.4 Bound Computation

We first find u_H from (35), and the output–adjoint $\psi_H \in X_H$ from

$$\nu(\psi_{H\,x}, v_x; 1) - (\psi_{H\,x}, v; u_H) = -\ell^O(v), \forall v \in X_H. \tag{37}$$

We then compute the hybrid flux $p_H \in Q$ from

$$b(v, p_H) = R_H(v) - \ell^O(v) - \nu(\psi_{H\,x}, v_x; 1) + (\psi_{H\,x}, v; u_H), \forall v \in \hat{X}_H, \tag{38}$$

where

$$R_H(v) \equiv \langle f, v\rangle - \nu(v_x, u_{H\,x}; 1) + \frac{1}{2}(v_x, u_H^2; 1) \tag{39}$$

is the residual. For future reference we note that, from (35), $R_H(v) = 0, \forall v \in X_H$. It can be readily shown that (38) has a unique solution.

Next, we find $\hat{e}_h \in \hat{X}_h$ from

$$2a_H(v, \hat{e}_h) = R_H(v) - \ell^O(v) - \nu(\psi_{H\,x}, v_x; 1) + (\psi_{H\,x}, v; u_H) - b(v, p_H), \forall v \in \hat{X}_h, \tag{40}$$

which we assume yields a unique solution; note that (40) is a symmetric *linear* equation. Finally, based on $\hat{e}_h$, we compute our lower estimator

$$s_{LO} = \ell^O(u_H) - a_H(\hat{e}_h, \hat{e}_h). \tag{41}$$

As for the Helmholtz problem, we may obtain a corresponding upper estimator for s_h simply by pursuing a lower estimator for $-\ell^O(u_h)$ [10, 11, 13]. (Note that for the nonlinear Burgers equation — in contrast to the linear Helmholtz equation — optimization of the estimators with respect to a scaling parameter κ is more difficult, and shall not be considered further here.)

3.5 Asymptotic Properties

We now define $e_h = u_h - u_H$, and $z_h = e_h - \hat{e}_h$. We then make the hypothesis that there exists an $H_0 > h$ such that, for all $H \leq H_0$,

$$B_H(z_h, z_h) \leq A(z_h, z_h), \tag{42}$$

or equivalently,

$$a_H(z_h, z_h) \geq 0. \tag{43}$$

This hypothesis is very similar to the corresponding hypothesis for the Helmholtz equation, however the theoretical issues are clearly more difficult since a_H and B_H now depend on u_H and ψ_H. In practice, however, we expect — if interpolation theory is an adequate guide — that (42) will be satisfied as soon as the working mesh $\mathcal{T}_H$ even roughly resolves the structure of u. Conversely, for very coarse — unresolved — meshes we do not expect that (42) will hold except in the uninteresting case in which ν is not small.

We now claim that, if (42) is satisfied, then $s_{LO} \leq \ell^O(u_h)$, as desired; a similar proof holds for the upper bound. We begin with several preliminary results that can be derived from the error equation for the nonlinear Burgers system,

$$\nu(v_x, e_{h\,x}; 1) - (v_x, e_h; u_H) - \frac{1}{2}(v_x, e_h^2; 1) = R_H(v), \forall v \in X_h. \tag{44}$$

In particular, if we consider $v \in X_H \subset X_h$, we obtain

$$\nu(v_x, e_{h\,x}; 1) - (v_x, e_h; u_H) - \frac{1}{2}(v_x, e_h^2; 1) = 0, \forall v \in X_H, \tag{45}$$

which is our "orthogonality" condition. Similarly, if we consider $v = e_h$, we obtain

$$\nu(e_{h\,x}, e_{h\,x}; 1) + \frac{1}{2}(e_h, e_h; u_{H\,x}) = R_H(e_h), \tag{46}$$

which is our "energy equality" [12].

Proceeding with the proof, we first note from (40) that

$$2a_H(e_h, \hat{e}_h) = R_H(e_h) - \ell^O(e_h) - \nu(\psi_{H\,x}, e_{h\,x}; 1) + (\psi_{H\,x}, e_h; u_H) \tag{47}$$

where we have evoked $e_h \in X_h$ and the definition of b, (13). We next note from our orthogonality relation (45) for $v = \psi_H$ that

$$-\nu(\psi_{H\,x}, e_{h\,x}; 1) + (\psi_{H\,x}, e_h; u_H) = -\frac{1}{2}(e_h, e_h; \psi_{H\,x}). \tag{48}$$

Inserting (48) and $R_H(e_h)$ from (46) into (47) we find that

$$2a_H(e_h, \hat{e}_h) = \nu(e_{h\,x}, e_{h\,x}; 1) + \frac{1}{2}(e_h, e_h; u_{H\,x}) - \ell^O(e_h) - \frac{1}{2}(e_h, e_h; \psi_{H\,x}), \quad (49)$$

or more succintly,

$$2a_H(e_h, \hat{e}_h) = a_H(e_h, e_h) - \ell^O(e_h). \quad (50)$$

The remainder of the proof follows exactly as in the case of the Helmholtz operator.

3.6 Numerical Results

We consider (28) with $\nu = 1/30$, $b_0 = 1$, $b_1 = 0$, and $f = 0$; the output chosen is the value of u at $x = 0.9$, a point within the boundary layer that forms at $x = 1$. The truth mesh is given by $h = 1/10{,}240$; the working mesh diameter varies from $H = h$ to $H = 10$. Note that on the coarsest working mesh there is only a single point within the boundary layer at $x = 1$. For each working mesh H we compute the lower (s_{LO}) and upper (s_{UP}) estimators, the results of which are presented in Figure 2 in the form of s_{LO}/s_h and s_{UP}/s_h.

The results are very similar to the corresponding results for the Helmholtz equation. First, we obtain bounds even for the largest H. As expected, $A(z_h, z_h)$ vanishes as H^2, while $B(z_h, z_h)$ vanishes as H^4; inspection of z_h indicates that, indeed, the error is predominated by high–wavenumber components. Second, the bounds converge quadratically; this follows directly from the observed behavior of $A(z_h, z_h)$ and $B(z_h, z_h)$ and the proof of Section 3.5. Third, there are no undetermined constants in the results presented in Figure 2; furthermore, given the hypothesis (42), and the assumption that s_h is sufficiently close to s, the bounds are rigorous — even for this nonlinear problem. Fourth, as for the Helmholtz problem, our bounds are directly for the quantity of interest.

Clearly, the Burgers equation (as well as the Helmholtz equation) is a good candidate for nonuniform meshes and adaptive refinement. This will be considered in a future paper based on the framework established here.

Acknowledgements

This work was supported by DARPA and ONR under Grant N00014-91-J-1889, AFOSR under Grant F49620-94-1-0121, and NASA under Grants NAG 1-1613, NAG 4-105, and NAG 1-1587.

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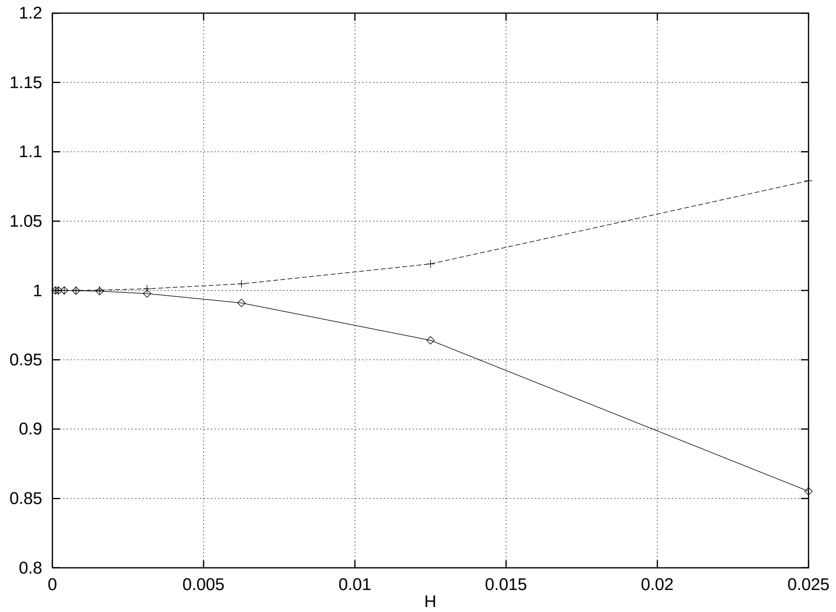

Figure 1: A plot of s_{LO}/s_h and s_{UP}/s_h as a function of H for the Helmholtz equation. We present results only for $H \leq 1/40$; for larger H, bounds are still obtained, however the bound gap is quite large.

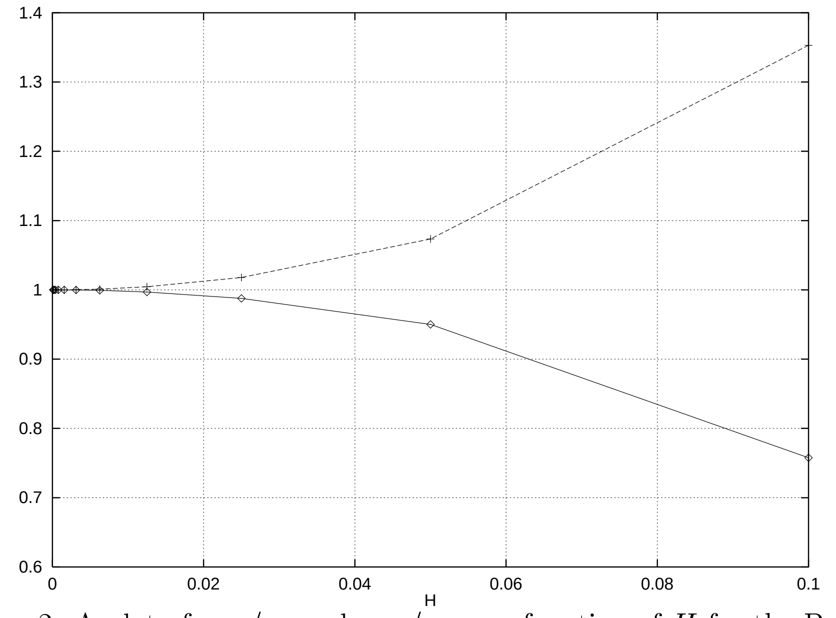

Figure 2: A plot of s_{LO}/s_h and s_{UP}/s_h as a function of H for the Burgers equation.